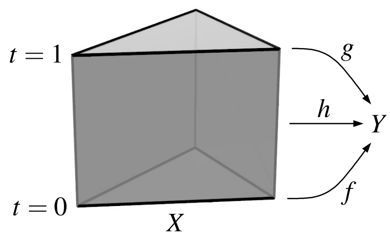
3. Chain Homotopies

In the previous chapter we have defined the simplicial homology groups of a simplicial complex. Due to their explicit construction using only finite-dimensional vector spaces, we saw that they can be computed with standard linear algebra techniques. However, the dimensions of these vector spaces are usually very large, and thus the brute-force algorithmic approach to determine the homology groups is often infeasible for real-world examples, and also unsuitable for proving theoretical results.

For example, at the moment we cannot even compute the simplicial homology of the disk D^α for a p-simplex α for all p: As α has $p+1$ vertices it has $\binom{p+1}{k+1}$ faces (i. e. subsets of cardinality $k+1$) of dimension $k \in \{0,\dots,p\}$, and thus the simplicial chain complex $C(D^\alpha)$ is already quite complicated. This might be surprising since the topological realization of D^α is just the disk D^p, which is of course a contractible space (i. e. homotopy equivalent to a point) and thus does not have any holes, i. e. it should have the (trivial) homology of a point. In fact, we will see in this chapter that this is true, but proving this statement directly by explicit computations with the simplicial chain complex would be messy.

But from this topological argument of contractibility we can already see a possible solution for our problems: We should talk about homotopy. In the same way as the fundamental group, the homology groups will be invariant under homotopy equivalences, so it will be a perfectly valid argument that the homology of a disk is trivial since it is homotopy equivalent to a point. However, homotopies are usually defined topologically in terms of continuous maps, so what does this concept mean in our current algebraic simplicial setting?

To see this, let us first look at the topological picture. As shown on the right, two continuous maps $f,g\colon X \to Y$ between topological spaces are called homotopic if there is a continuous *homotopy map* $h\colon X\times[0,1]\to Y,\ (x,t)\mapsto h(x,t)$ on the product space $X\times[0,1]$ (which is geometrically a prism with base X) such that $h(x,0)=f(x)$ and $h(x,1)=g(x)$ for all $x\in X$. Intuitively, this means that f can be deformed continuously into g.



To translate this concept into algebra, let us first of all simplify the picture above by just looking at the prism, ignoring the maps to Y for a moment – our constructions will be functorial, so once we have described the prism it will be no problem for us to map this situation to some target space. Next, let us pass to the simplicial picture and regard X as the topological realization of a simplicial p-chain, so that the prism $X\times[0,1]$ should be the topological realization of a $(p+1)$-chain. In the above picture, we have drawn this situation for a single 2-simplex α: We can imagine $f(\alpha)$ as the bottom face of the prism, $g(\alpha)$ as its top face, and $h(\alpha)$ as the whole prism (let us postpone for a moment how we can realize this prism as a simplicial $(p+1)$-chain; we will address this problem in Construction 3.6).

Now let us phrase this geometric situation in terms of boundary maps: The boundary of the prism $\partial h(\alpha)$ consists of its bottom face $f(\alpha)$, its top face $g(\alpha)$, and its sides. The key observation is now that these sides are just another prism whose base is the boundary $\partial\alpha$, so they can be written as $h(\partial\alpha)$. In short, $\partial h(\alpha)$ is a combination of $f(\alpha)$, $g(\alpha)$, and $h(\partial\alpha)$. To get the signs right in this linear combination, note first that $f(\alpha)$ and $g(\alpha)$ should clearly have opposite orientations as they sit on opposite faces of the prism. But the remaining signs are just conventional as they can be absorbed by the precise construction of the homotopy map h (note that in $\partial h(\alpha)$ it takes prisms over p-chains and in $h(\partial\alpha)$ over $(p-1)$-chains, so these are actually two different prism maps that can also be given independent signs). The standard convention is to just take the sign to be positive for both $\partial h(\alpha)$ and $h(\partial\alpha)$, so we arrive at the equation

$$\partial h(\alpha) + h(\partial\alpha) = g(\alpha) - f(\alpha).$$

In fact, this is the purely algebraic description of homotopy that we were looking for. It makes sense to formulate it in terms of arbitrary chain complexes in this way, so let us write down the corresponding definition.

Definition 3.1 (Chain homotopies)**.** Two morphisms $f,g\colon C \to D$ of chain complexes as shown in the diagram below are called **(chain) homotopic** if there is a collection $h = (h_p)_{p\in\mathbb{Z}}$ of linear maps $h_p\colon C_p \to D_{p+1}$ with $\partial_{p+1}\circ h_p + h_{p-1}\circ\partial_p = g_p - f_p$ for all $p\in\mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{p+1} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & C_p \arrow[r, "\partial_p"] \arrow[dl, "h_p"'] \arrow[d, shift right, "f_p"'] \arrow[d, shift left, "g_p"] & C_{p-1} \arrow[r] \arrow[dl, "h_{p-1}"] \arrow[d, shift right] \arrow[d, shift left] & \cdots \\
\cdots \arrow[r] & D_{p+1} \arrow[r, "\partial_{p+1}"'] & D_p \arrow[r] & D_{p-1} \arrow[r] & \cdots
\end{tikzcd}$$

The collection h is then called a **homotopy** or **prism map** between f and g. To simplify notations, we will usually write all maps h_p just as h, so that the homotopy equation becomes $\partial\circ h + h\circ\partial = g - f$.

Remark 3.2.

(a) It is checked immediately that homotopy is an equivalence relation on the set of all morphisms between given chain complexes.

(b) Although a homotopy h between two morphisms $f,g\colon C\to D$ of chain complexes maps the vector spaces of C to the vector spaces of D, it is clearly not a morphism from C to D in the sense of Definition 2.8 (b):

- It shifts the dimension index, i. e. does not map C_p to D_p, but to D_{p+1}.
- It does not form a commutative diagram of maps in any way, but instead satisfies the homotopy equation $\partial\circ h + h\circ\partial = g - f$.
- It does not induce maps on homology between $H_p(C)$ and $H_{p+1}(D)$.

Instead, as explained above its importance lies in the fact that its existence proves that f and g are homotopic, and thus define the same map on homology:

Proposition 3.3. *Two homotopic morphisms $f,g\colon C\to D$ of chain complexes induce the same map on homology, i. e. we have $f_p = g_p$ as maps from $H_p(C)$ to $H_p(D)$ for all $p\in\mathbb{Z}$.*

Proof. Let $c\in H_p(C)$ be a cycle, i. e. $\partial c = 0$. For a homotopy h between f and g,

$$f(c) - g(c) = \partial h(c) + h(\partial c) = \partial h(c)$$

is then a boundary, hence zero in $H_p(D)$. □

As in topology, this leads immediately to the definition of homotopy equivalence.

Definition 3.4 (Homotopy equivalence)**.** A morphism $f\colon C\to D$ of chain complexes is a **homotopy equivalence** if there is a morphism $g\colon D\to C$ such that $g\circ f$ is homotopic to id_C and $f\circ g$ is homotopic to id_D. In this case, g is called a **homotopy inverse** of f.

Corollary 3.5. *Any homotopy equivalence $f\colon C\to D$ of chain complexes induces isomorphisms $f_p\colon H_p(C)\to H_p(D)$ between their homology groups for all $p\in\mathbb{Z}$.*

Proof. Let g be a homotopy inverse of f. As homology is a functor by Lemma 2.17, we already know that there are induced maps $f_p\colon H_p(C)\to H_p(D)$ and $g_p\colon H_p(D)\to H_p(C)$ for all $p\in\mathbb{Z}$ whose compositions are $g_p\circ f_p\colon H_p(C)\to H_p(C)$ and $f_p\circ g_p\colon H_p(D)\to H_p(D)$. But these maps are just the identity by Proposition 3.3 as $g\circ f$ and $f\circ g$ are homotopic to the identity by assumption. Hence, f_p is an isomorphism with inverse g_p. □

So far, our setting for homotopy was purely algebraic. To apply this to simplicial homology, we now have to construct prisms in the category of simplicial complexes. Note that it is not immediately obvious how to do this since the product of a simplex with an interval – as in the picture in the introduction to this chapter – is not directly a simplex again. However, there are ways to subdivide this space into simplices, and this will be sufficient for our purposes. We will see one possibility for such a subdivision now; another one will be considered in Construction 7.2.

Construction 3.6 (Prism construction)**.** To see how to construct the prism over a simplicial complex, let us first consider as an example the $(p-1)$-simplex $\alpha = \{1,2,3,\dots,p\}$ with its standard geometric realization $|\alpha| \subset \mathbb{R}^{\{1,\dots,p\}}$. By Construction 1.12 and Proposition 1.15, the prism over α is then given in barycentric coordinates for α as

$$P = \{(x_1,\dots,x_p,t) : x_1,\dots,x_p \in \mathbb{R}_{\geq 0}, x_1+\cdots+x_p = 1, 0 \leq t \leq 1\} \quad \subset \mathbb{R}^{\{1,\dots,p\}} \times \mathbb{R}.$$

It has $2p$ vertices $(e_i,0)$ and $(e_i,1)$ for $i = 1,\dots,p$, which we label by i and i', respectively; the case $p=3$ is shown in the picture below on the left. The idea to subdivide it into simplices is to consider the p inequalities

$$0 \leq x_p \leq x_{p-1}+x_p \leq x_{p-2}+x_{p-1}+x_p \leq \cdots \leq x_1+\cdots+x_p = 1$$

and subdivide P according to where the number $t \in [0,1]$ fits in. For $k = 1,\dots,p$ let us denote by $P_k \subset P$ the subset where t fits into the i-th inequality counted from the right, i. e. where

$$0 \leq x_p \leq \cdots \leq x_{k-1}+\cdots+x_p \leq t \leq x_k+\cdots+x_p \leq \cdots \leq x_1+\cdots+x_p = 1. \qquad (*)$$

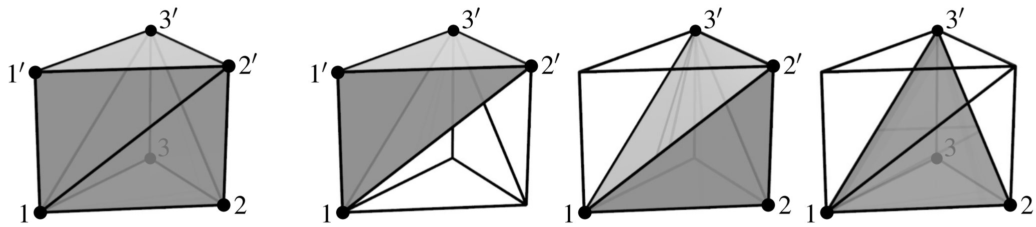
One can show that this is actually the geometric realization of a p-simplex, with vertices given by the points where only one of the $p+1$ inequalities in $(*)$ is not an equality. Now for $i = 1,\dots,p+1$ the point where only the i-th inequality (again counted from the right) is not an equality is given by

$$0 = \cdots = t = \cdots = x_{i-1}+\cdots+x_p \leq x_i+\cdots+x_p = \cdots = 1$$

for $i \leq k$, which is just $(e_i,0)$, and by

$$0 = \cdots = x_{i-2}+\cdots+x_p \leq x_{i-1}+\cdots+x_p = \cdots = t = \cdots = 1$$

for $i > k$, which is $(e_{i-1},1)$. Hence, P_k is the geometric realization (in our chosen coordinates) of the simplex $\{1,\dots,k,k',\dots,p\}$. The picture below on the right shows how our prism P in the case $p=3$ is subdivided into the simplices $\{1,1',2',3'\}$, $\{1,2,2',3'\}$, and $\{1,2,3,3'\}$.



Luckily, we do not need a full exact proof that this procedure really works and extends to simplicial complexes. We just take the computations above as a motivation for the following simplicial definition of the prism map, for which we then do prove rigorously that it has the desired properties. Note that in order for this to work we have to choose an order on the vertices as we needed this to set up the inequalities $(*)$ above.

Definition 3.7 (Simplicial prism map)**.** Let S be a simplicial complex, for which we fix an order on its set of vertices $V(S)$.

(a) The **prism** over S is defined to be the simplicial complex

$$P(S) := \big\{\{i_0,\dots,i_{k-1},i'_k,\dots,i'_p\} : \\ k \leq p \text{ and } \{i_0,\dots,i_p\} \in S \text{ with } i_0 < \cdots < i_{k-1} \leq i_k < \cdots < i_p\big\},$$

with vertices labeled i and i' for $i \in V(S)$. It comes together with inclusion morphisms $f,g\colon S \to P(S)$ of the bottom resp. top face given by $f(i) = i$ and $g(i) = i'$ for all $i \in V(S)$.

(b) The associated **prism map** for $p \in \mathbb{Z}$ is the linear map $h_p \colon C_p(S) \to C_{p+1}(P(S))$ given on p-simplices $\{i_0,\dots,i_p\} \in S$ with $i_0 < \cdots < i_p$ by

$$h_p([i_0,\dots,i_p]) = \sum_{k=0}^{p} (-1)^k \cdot [i_0,\dots,i_k,i'_k,\dots,i'_p] \quad \in C_{p+1}(P(S)).$$

Lemma 3.8 (Simplicial homotopies)**.** *Let S be a simplicial complex with a fixed ordering of its set of vertices $V(S)$. With notations as in Definition 3.7, the prism map h is a homotopy between the morphisms of simplicial chain complexes induced by the bottom and top face inclusion of S in $P(S)$, i. e. we have*

$$\partial \circ h + h \circ \partial = g_* - f_*$$

as morphisms $C(S) \to C(P(S))$ of chain complexes. In particular, f_* and g_* agree as linear maps $H_p(S) \to H_p(P(S))$ on homology for all $p \in \mathbb{Z}$ by Proposition 3.3.

Proof. This is just a computation, and by our motivation from Construction 3.6 it should not be surprising that it works out: For any p-simplex $\{i_0,\dots,i_p\} \in S$ with $i_0 < \cdots < i_p$ we have

$$\begin{aligned}\partial h([i_0,\dots,i_p]) &= \partial \sum_{k=0}^{p} (-1)^k \cdot [i_0,\dots,i_k,i'_k,\dots,i'_p] \\ &= \sum_{k,l:\, l\le k} (-1)^{k+l} \cdot [i_0,\dots,\widehat{i_l},\dots,i_k,i'_k,\dots,i'_p] + \sum_{k,l:\, l\ge k} (-1)^{k+l+1} \cdot [i_0,\dots,i_k,i'_k,\dots,\widehat{i'_l},\dots,i'_p]\end{aligned}$$

and

$$\begin{aligned}h(\partial[i_0,\dots,i_p]) &= h\left(\sum_{l=0}^{p} (-1)^l \cdot [i_0,\dots,\widehat{i_l},\dots,i_p]\right) \\ &= \sum_{k,l:\, k<l} (-1)^{k+l} \cdot [i_0,\dots,i_k,i'_k,\dots,\widehat{i'_l},\dots,i'_p] + \sum_{k,l:\, k>l} (-1)^{k+l-1} \cdot [i_0,\dots,\widehat{i_l},\dots,i_k,i'_k,\dots,i'_p].\end{aligned}$$

Note that in the sum of these two expressions all summands cancel out except the ones for $k = l$ (which are only present in the first expression), and thus we get as desired

$$\begin{aligned}\partial h([i_0,\dots,i_p]) + h(\partial[i_0,\dots,i_p]) &= \sum_{k=0}^{p} [i_0,\dots,i_{k-1},i'_k,\dots,i'_p] - \sum_{k=0}^{p} [i_0,\dots,i_k,i'_{k+1},\dots,i'_p] \\ &= [i'_0,\dots,i'_p] - [i_0,\dots,i_p] \\ &= g_*([i_0,\dots,i_p]) - f_*([i_0,\dots,i_p]).\end{aligned}$$

□

As a reality check, Lemma 3.8 is just the algebraic simplicial manifestation of the obvious topological statement that the bottom face of a prism can be deformed continuously to the top face by moving upwards. In particular, so far we have just considered the prism over a simplicial complex itself. Just as in the topological setting, to obtain useful results we now have to map this prism to another simplicial complex and use that the restrictions to the bottom and top face of the prism are homotopic.

In topology, the simplest application of this type is arguably the statement that a star-shaped set is contractible. Recall that a topological subspace $X \subset \mathbb{R}^n$ is called star-shaped with respect to $a \in X$ if for every point $x \in X$ the line segment connecting a and x lies in X as well. We can then construct a homotopy equivalence between X and the one-point space $\{a\}$ by moving every point $x \in X$ on a straight line to a. In fact, the same idea also works in the simplicial setting as follows.

Definition 3.9 (Star-shaped simplicial complexes)**.** A simplicial complex S is called **star-shaped** if there is a vertex $i \in V(S)$ such that $\alpha \cup \{i\} \in S$ for all $\alpha \in S$. We then say that S is star-shaped with respect to i.

Example 3.10. Let $\alpha = \{i_0,\dots,i_p\}$ be an arbitrary p-simplex.

(a) The disk D^α is clearly star-shaped (with respect to any of its vertices). In contrast, the sphere S^α is not star-shaped: For any $k \in \{0,\dots,p\}$ we have $\{i_0,\dots,\widehat{i_k},\dots,i_p\} \in S^\alpha$, but $\{i_0,\dots,\widehat{i_k},\dots,i_p\} \cup \{i_k\} \notin S^\alpha$.

(b) The barycentric subdivision $\operatorname{Sd}(D^\alpha)$ of the disk D^α as in Definition 1.20 is star-shaped with respect to its barycenter $\{i_0,\dots,i_p\}$ (but not with respect to any other of its vertices).

Proposition 3.11 (Betti numbers of star-shaped simplicial complexes)**.** *Any star-shaped simplicial complex S has the homology of a point, i. e. Betti numbers $b_0(S)=1$ and $b_p(S)=0$ for all $p\neq 0$. We will say that a simplicial complex with these Betti numbers has **trivial homology**.*

Proof. Let S be star-shaped with respect to $j\in S$. Consider the morphism $h\colon P(S)\to S$ given by $h(i)=i$ and $h(i')=j$ for all $i\in V(S)$, i. e. h is the identity on the bottom face and the constant map j on the top face of the prism $P(S)$. To see that this is in fact well-defined we have to show that the image of any simplex $\{i_0,\dots,i_{k-1},i'_k,\dots,i'_p\}\in P(S)$ lies in S. This is obvious if $k=0$ or $k=p+1$. For $0<k<p+1$ note that $\{i_0,\dots,i_p\}\in S$ by definition of $P(S)$, hence also $\{i_0,\dots,i_{k-1}\}\in S$, and thus

$$h(\{i_0,\dots,i_{k-1},i'_k,\dots,i'_p\})=\{i_0,\dots,i_{k-1},j,\dots,j\}=\{i_0,\dots,i_{k-1}\}\cup\{j\}\in S$$

since S is star-shaped with respect to j. Thus, the map $h\colon P(S)\to S$ is in fact well-defined.

Now let $f,g\colon S\to P(S)$ be the inclusion of the bottom and top face as in Definition 3.7 (a). Then $h\circ f=\mathrm{id}_S$, and $h\circ g$ is the constant map j. Hence, for the pushforward maps $H_p(S)\to H_p(S)$ by these compositions we have for all $p\in\mathbb{Z}$

$$b_p(S)=\operatorname{rk}\mathrm{id}_{H_p(S)}=\operatorname{rk}(h_*\circ f_*)_p\overset{3.8}{=}\operatorname{rk}(h_*\circ g_*)_p.$$

But the pushforward $(h_*\circ g_*)_p\colon H_p(S)\to H_p(S)$ by the constant map j is just zero for $p\neq 0$ and $\alpha\mapsto(\deg\alpha)\cdot[j]$ of rank 1 for $p=0$ (with the degree as in Lemma 2.22), so the result follows. □

Example 3.12 (Simplicial homology of disks)**.** By Example 3.10, Proposition 3.11 implies that for any p-simplex α the disk D^α and the subdivided disk $\operatorname{Sd}(D^\alpha)$ have trivial homology – as already mentioned in the introduction to this chapter and expected from topology (since the topological realization of both simplicial complexes is just the contractible disk D^p). 05

Example 3.13 (Simplicial homology of spheres)**.** Using Example 3.12, we can now also derive the simplicial homology of a p-dimensional sphere S^α for a $(p+1)$-simplex $\alpha=\{i_0,\dots,i_{p+1}\}$: Note that $D^\alpha=S^\alpha\cup\{\alpha\}$, so

$$\dim C_k(S^\alpha)=\begin{cases}\dim C_k(D^\alpha) & \text{if } k\neq p+1,\\ \dim C_k(D^\alpha)-1 & \text{if } k=p+1.\end{cases}\qquad (*)$$

A similar result holds for the simplicial homology: As the definition of $H_k(S^\alpha)$ only uses the part $C_{k+1}(S^\alpha)\to C_k(S^\alpha)\to C_{k-1}(S^\alpha)$ of the simplicial chain complex, and this agrees with that of D^α for $k\notin\{p,p+1,p+2\}$ by $(*)$, the resulting homology groups must then be the same. But S^α does not even have simplices of dimensions $p+1$ and $p+2$, so these Betti numbers are clearly zero – just as for D^α. Hence, the only Betti number of S^α that can differ from that of D^α is b_p, which means that we can obtain it easily from an Euler characteristic computation: By Lemma 2.25 we have

$$(-1)^p(b_p(S^\alpha)-b_p(D^\alpha))=\chi(S^\alpha)-\chi(D^\alpha)=(-1)^{p+1}\underbrace{(\dim C_{p+1}(S^\alpha)-\dim C_{p+1}(D^\alpha))}_{=-1\text{ by }(*)},$$

and thus $b_p(S^\alpha)=b_p(D^\alpha)+1$. In summary, this means that for $p>0$ we have

$$b_k(S^\alpha)=\begin{cases}1 & \text{for } k=0 \text{ or } k=p,\\ 0 & \text{otherwise.}\end{cases}$$

The case $p=0$ is special and gives $b_0(S^\alpha)=2$ and $b_k(S^\alpha)=0$ otherwise by the same argument, but this should not be surprising since then $S^\alpha=\{\{i_0\},\{i_1\}\}$ is just a disjoint union of two points.

Staying in the case $p>0$, it is also easy to write down a generator of the one-dimensional vector space $H_p(S^\alpha)$: The p-chain

$$\partial[i_0,\dots,i_{p+1}]=\sum_{k=0}^{p+1}(-1)^k\cdot[i_0,\dots,\widehat{i_k},\dots,i_{p+1}]\quad\in C_p(S^\alpha)$$

is closed by Lemma 2.12 and not a boundary as S^α has no $(p+1)$-simplices, so it must be a generator. Geometrically, it just corresponds to "the sphere S^α itself" considered as a p-cycle in S^α.

Note however that this generator is not entirely natural since a change of orientation of the vertices would replace it by its negative. This happens for example for the isomorphism $S^\alpha \to S^\alpha$ exchanging two of the vertices i_k and i_l with $k < l$, which in the standard geometric realization corresponds to a reflection at the hyperplane spanned by $e_0, \dots, \widehat{e_k}, \dots, \widehat{e_l}, \dots, e_p$, and $e_k + e_l$.

Exercise 3.14. Let $f, g \colon S \to T$ be two morphisms of simplicial complexes with $f(\alpha) \cup g(\alpha) \in T$ for any $\alpha \in S$. Show that the pushforwards f_* and g_* agree as maps in homology $H_p(S) \to H_p(T)$ for all $p \in \mathbb{Z}$.