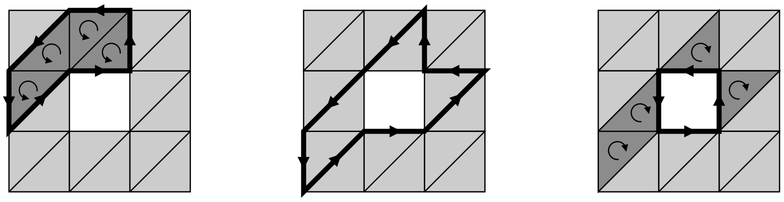
2. The Construction of Simplicial Homology

In the last chapter we have introduced simplicial complexes as a combinatorial way to describe certain topological spaces. Let us now proceed to construct the *simplicial homology groups* which will be able to detect "holes" in such spaces. It should be said clearly that this construction (and in fact everything we do from now on until Chapter 5) is entirely combinatorial resp. algebraic and does not use topology in any way. However, the questions we ask and the methods we use will strongly be motivated by topology, so the pictures we draw will often look very topological. In fact, we will see in Chapter 8 that simplicial complexes with homeomorphic topological realizations will have isomorphic simplicial homology groups – but this is not at all obvious from the definitions and will require substantial effort to prove.

As already mentioned in the introduction, the idea to detect holes is to consider "closed cycles modulo boundaries". Consider for example the 2-dimensional simplicial complex realizing an annulus shown in the picture below. In the left picture, the thick lines show a loop (which we will call a closed 1-chain, or a 1-cycle) which is the boundary of the two-dimensional dark region (which we will call a 2-chain). Hence this loop will vanish in homology when we consider cycles modulo boundaries, which means intuitively that this loop does not detect a hole in the simplicial complex. On the other hand, the picture in the middle shows a 1-cycle which cannot be written as a boundary, so it runs around a hole and will be a non-trivial element in homology. Finally, the same holds for the 1-cycle in the picture on the right, but the boundary of the dark region is the difference of it and the 1-cycle from the middle picture, so these two 1-cycles are the same modulo boundaries. Hence they will represent the same homology class, corresponding to the intuitive statement that they detect the same hole in the space.



You have probably noticed that the above explanation implicitly used *orientations*, at the latest when we said that the boundary of the dark region in the right picture is the *difference* of two 1-cycles. This is more or less forced on us if we want the spaces describing homology to be groups, and corresponds to the fact that a loop in the fundamental group of a topological space becomes its inverse if we reverse its orientation. Formally, an orientation of a simplex will be an ordering of its vertices modulo even permutations. So for 1-simplices it just is an ordering of its vertices (we have indicated this above by the arrows on the loops), and for 2-simplices it is a cyclic ordering (indicated by the circular arrows above). In higher dimensions, it would be harder to visualize.

Also, to obtain groups we need to be able to add and subtract chains, and thus we have to consider sums of oriented simplices at least with integer coefficients. This is how homology groups are usually constructed in the literature, but for our purposes it is more convenient to allow coefficients in a chosen ground field: Firstly, when we move on to analysis later on we will have to pass to real coefficients anyway, and secondly, this makes all our spaces into vector spaces and thus accessible to linear algebra. Hence, our homology groups will actually be "homology vector spaces", but we will stick to the term "homology group" as this is the common name in the literature.

Let us now start with the formal definition of chains, following the ideas above. From now on, let K always denote a fixed ground field.

Definition 2.1 (Ordered simplicial chains)**.** Let S be a simplicial complex, and $p \in \mathbb{N}$.

(a) An **ordered p-simplex** is just an arbitrary $(p+1)$-tuple $(i_0,\dots,i_p)$. We say that it is an ordered p-simplex in S if $\{i_0,\dots,i_p\} \in S$. Note that we do not require the vertices $i_0,\dots,i_p$ to be distinct.

(b) We denote by $C_p^{\text{ord}}(S)$ the K-vector space with basis given by all ordered p-simplices in S. Its elements (which are then K-linear combinations of ordered p-simplices in S) will be called **ordered simplicial p-chains** in S.

For our purposes, it will be convenient to define the vector space $C_p^{\text{ord}}(S)$ for all $p \in \mathbb{Z}$, so we set $C_p^{\text{ord}}(S) := 0$ for all $p < 0$. If we want to indicate the ground field in the notation we will write $C_p^{\text{ord}}(S)$ also as $C_p^{\text{ord}}(S,K)$.

Definition 2.2 (Oriented simplicial chains)**.** Again let S be a simplicial complex, and let $p \in \mathbb{N}$. We denote by $C_p(S)$ the quotient vector space of $C_p^{\text{ord}}(S)$ modulo the subspace generated by:

- all ordered p-simplices $(i_0,\dots,i_p)$ for which $i_0,\dots,i_p$ are not pairwise distinct; and
- all ordered p-chains $(i_0,\dots,i_p) - \operatorname{sign}(\pi)\cdot(i_{\pi(0)},\dots,i_{\pi(p)})$ for permutations π of $\{0,\dots,p\}$.

The class of an ordered p-simplex $(i_0,\dots,i_p)$ in $C_p(S)$ is called an **oriented p-simplex** in S and will be written as $[i_0,\dots,i_p]$. Elements of $C_p(S)$ will be called **(oriented) simplicial p-chains** in S.

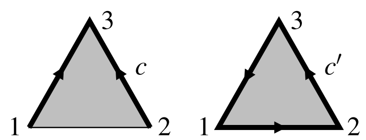
As before, we set $C_p(S) = 0$ for all $p < 0$, and write $C_p(S)$ as $C_p(S,K)$ if we want to indicate the ground field in the notation.

Remark 2.3. Let S be a simplicial complex, and let $p \in \mathbb{Z}$.

(a) By construction, $C_p(S)$ is a vector space whose dimension is the number of p-simplices in S. For any such p-simplex $\{i_0,\dots,i_p\}$ there is a corresponding basis vector $[i_0,\dots,i_p] \in C_p(S)$ that changes its sign if we apply an odd permutation to the vertices. In particular, $C_p(S)$ will be the zero vector space if $p < 0$ or $p > \dim S$. Oriented simplices with repeated vertices are allowed, but equal to 0 in $C_p(S)$ by definition.

(b) The vector spaces $C_p^{\text{ord}}(S)$ are still finite-dimensional, but much bigger since permutations of the vertices in an ordered simplex lead to new ordered simplices. In fact, as repeated vertices are allowed in ordered simplices, $C_p^{\text{ord}}(S)$ will even be non-zero for arbitrary $p \in \mathbb{N}$ unless S is empty. Hence, it would be computationally much more expensive to work with $C_p^{\text{ord}}(S)$ than with $C_p(S)$. Fortunately, working with $C_p(S)$ will be fine for us for the moment; we will come back to the ordered simplicial chains in Construction 4.23.

Example 2.4. In the simplicial disk $D^{\{1,2,3\}}$, the picture on the right shows two oriented simplicial 1-chains: $c = [1,3]+[2,3]$ and $c' = [1,2]+[2,3]+[3,1]$. Note that we could also write c' as $[1,2]-[1,3]+[2,3]$, and that $c+c' = [1,2]+2\,[2,3]$. In particular, in case of the ground field $\mathbb{Z}_2$ we would have $c+c' = [1,2]$.



Let us now construct the boundary map for oriented chains. By definition, the boundary of a p-simplex is always meant to be the chain of its $(p-1)$-dimensional faces (so no faces of smaller dimensions are involved). This makes the following definition straightforward, we just have to get the orientations right:

Definition 2.5 (Boundary map for simplicial chains)**.** Let S be a simplicial complex.

(a) For $p > 0$, the **boundary** of an oriented p-simplex $[i_0,\dots,i_p]$ in S is the $(p-1)$-chain

$$\partial_p[i_0,\dots,i_p] := \sum_{k=0}^{p} (-1)^k \cdot [i_0,\dots,\widehat{i_k},\dots,i_p] \quad \in C_{p-1}(S),$$

where the notation $\widehat{i_k}$ means that this vertex is left out.

By linearity, we extend this to a linear **boundary map** $\partial_p\colon C_p(S) \to C_{p-1}(S)$. To extend it to all integers, we define $\partial_p\colon C_p(S) \to C_{p-1}(S)$ to be the zero map for $p \le 0$. Often we will drop the index and write the boundary map ∂_p just as ∂.

(b) A p-chain $c \in C_p(S)$ is called **closed**, or a p**-cycle**, if $\partial c = 0$. It is called a **boundary** if $c = \partial c'$ for some $c' \in C_{p+1}(S)$.

Remark 2.6. Whenever we define something on oriented simplices such as in Definition 2.5 (a), we have of course to check that it is well-defined, i. e. that the definition is compatible with vertex permutations. To do this for Definition 2.5 (a), we write its right hand side for given $0 \le a < b \le p$ as

$$(-1)^a \cdot \underbrace{[\dots,\widehat{i_a},\dots,i_b,\dots]}_{\text{(A)}} + (-1)^b \cdot \underbrace{[\dots,i_a,\dots,\widehat{i_b},\dots]}_{\text{(B)}} + \sum_{k \neq a,b} (-1)^k \cdot \underbrace{[\dots,\widehat{i_k},\dots]}_{\text{(C)}}. \qquad (*)$$

Now consider the two cases of Definition 2.2:

- If $i_a = i_b$ then (C) $= 0$, and (B) is obtained from (A) by moving i_b to the left $b-a-1$ places, giving a sign of $(-1)^{b-a-1}$. Hence, $(*) = ((-1)^a + (-1)^b(-1)^{b-a-1})\,\text{(A)} + \text{(C)} = 0$.
- Exchanging i_a and i_b in $[i_0,\dots,i_p]$ transforms (C) to $-$(C), (A) (again by moving the b-th position $b-a-1$ places to the left) to $(-1)^{b-a-1}$(B), and similarly (B) to $(-1)^{b-a-1}$(A). So $(*)$ becomes
$$(-1)^a(-1)^{b-a-1}\text{(B)} + (-1)^b(-1)^{b-a-1}\text{(A)} - \text{(C)} = -(*),$$
i. e. $(*)$ is multiplied by -1, which is exactly the sign of the transposition exchanging i_a and i_b. The general statement then follows since every permutation is a composition of transpositions.

Note that the sign $(-1)^k$ in Definition 2.5 (a) was important for this calculation to work out.

As in this example, checking well-definedness for constructions involving oriented simplices is usually just a straightforward computation. In the future, we will mostly omit such calculations.

Example 2.7. In Example 2.4, we have

$$\begin{aligned} \partial c &= ([3]-[1]) + ([3]-[2]) = 2\,[3]-[1]-[2] \\ \text{and} \quad \partial c' &= ([2]-[1]) + ([3]-[2]) + ([1]-[3]) = 0 \end{aligned}$$

in $C_0(D^{\{1,2,3\}})$. Hence, c' is a 1-cycle in $D^{\{1,2,3\}}$. In fact, it is also a boundary since

$$\partial[1,2,3] = [2,3]-[1,3]+[1,2] = c'.$$

Intuitively, this just says that c' describes a closed loop which is the boundary of the triangle.

With our constructions, we have now defined a sequence of vector spaces $C_p(S)$ for $p \in \mathbb{Z}$ for every simplicial complex S, with linear maps $\partial_p \colon C_p(S) \to C_{p-1}(S)$ between them. We will see many more sequences of similar type later on, so let us formalize this concept.

Definition 2.8 (Sequences of vector spaces)**.**

(a) A **sequence of vector spaces** (C,∂) is a collection of vector spaces $C = (C_p)_{p\in\mathbb{Z}}$ and linear maps $\partial = (\partial_p)_{p\in\mathbb{Z}}$ with $\partial_p \colon C_p \to C_{p-1}$ for all $p \in \mathbb{Z}$, i. e. we can write this as

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots.$$

We will usually write such a sequence just as C if ∂ is clear from the context.

(b) For two such sequences (C,∂) and (D,∂'), a **morphism** f between them is given by a collection of linear maps $f_p \colon C_p \to D_p$ for all $p \in \mathbb{Z}$ such that the diagram

$$\begin{tikzcd} \cdots \arrow[r] & C_{p+1} \arrow[r, "\partial_{p+1}"] \arrow[d, "f_{p+1}"] & C_p \arrow[r, "\partial_p"] \arrow[d, "f_p"] & C_{p-1} \arrow[r] \arrow[d, "f_{p-1}"] & \cdots \\ \cdots \arrow[r] & D_{p+1} \arrow[r, "\partial'_{p+1}"] & D_p \arrow[r, "\partial'_p"] & D_{p-1} \arrow[r] & \cdots \end{tikzcd}$$

is **commutative**. This means that any two paths along arrows with the same source and target in this diagram are the same map, i. e. in this case that $f_{p-1} \circ \partial_p = \partial'_p \circ f_p$ for all $p \in \mathbb{Z}$.

Often, the linear maps of the sequences C and D will be boundary maps of some kind, and thus all be denoted by ∂. Moreover, as for ∂ we will usually omit the index from f, and thus often write the commutativity condition just as $f \circ \partial = \partial \circ f$ (which also explains the name "commutativity").

This makes sequences of vector spaces into a category, which we will denote by Seq.

Construction 2.9 (Pushforwards and the simplicial chain functor)**.** Rephrasing our constructions in Definition 2.2 and 2.5 in the language of Definition 2.8, we have now associated to every simplicial complex S a sequence of vector spaces

$$\cdots \longrightarrow C_{p+1}(S) \xrightarrow{\partial_{p+1}} C_p(S) \xrightarrow{\partial_p} C_{p-1}(S) \longrightarrow \cdots,$$

which we will just denote by $C(S)$. Let us make this assignment $S \mapsto C(S)$ into a functor from Simp to Seq by also assigning to a morphism $f\colon S \to T$ of simplicial complexes an induced morphism $C(S) \to C(T)$ of sequences of vector spaces. The official name of this induced morphism in the language of Notation 1.32 would be $C(f)$, but it is common to denote it by $f_*\colon C(S) \to C(T)$ and call it the **pushforward** by f (referring to the fact that it moves a simplicial chain on S to one on T, i. e. "forward" in the same direction as the morphism f). Its construction is straightforward: We just define it by

$$f_*[i_0,\dots,i_p] := [f(i_0),\dots,f(i_p)] \quad \text{for all } [i_0,\dots,i_p] \in C_p(S),$$

extended linearly to a map $C_p(S) \to C_p(T)$ for all $p \in \mathbb{Z}$. Intuitively, if we think of a p-chain as a "p-dimensional object" in the topological space $|S|$, its pushforward can be thought of as its image under the continuous map $|f|\colon |S| \to |T|$.

It is immediate to see that this is well-defined in the sense of Remark 2.6 and commutes with the boundary maps as in Definition 2.8, so we have actually constructed a pushforward morphism of sequences of vector spaces $f_*\colon C(S) \to C(T)$. Moreover, it is obvious that the assignments $S \mapsto C(S)$ and $f \mapsto f_*$ give rise to a functor Simp $\to$ Seq, i. e. that they are compatible with compositions and the identity. We call it the *simplicial chain functor*.

Example 2.10. Consider again the morphism f of Example 1.28 (c) sending the torus S of Example 1.17 (d) with vertices labeled $1,\dots,9$ to the sphere $S^{\{1,2,3\}}$ by mapping the vertex $3i+j$ to j for all $i = \{0,1,2\}$ and $j \in \{1,2,3\}$. For this map, we have e. g. for the 2-chain $[1,2,5] \in C_2(S)$

$$\begin{aligned} f_*\partial[1,2,5] &= f_*([2,5]-[1,5]+[1,2]) = [2,2]-[1,2]+[1,2] = 0 \\ \text{and also} \quad \partial f_*\partial[1,2,5] &= \partial[1,2,2] = 0 \end{aligned}$$

in $C_1(S^{\{1,2,3\}})$, verifying in this example that f_* is a morphism of sequences. Note that this makes use of the fact that oriented simplices with repeated vertices are zero.

Remark 2.11. If we have a sequence of vector spaces (C,∂), the most natural first question is probably how long non-zero elements can "survive" in it, i. e. given a non-zero element $c \in C_p$ for some $p \in \mathbb{Z}$, and forming $\partial c \in C_{p-1}$, $\partial^2 c := \partial\partial c \in C_{p-2}$, and so on, what is the first $k \in \mathbb{N}$ such that $\partial^k c = 0$ in C_{p-k}? We will study this question in general in Proposition 9.13 when discussing persistent homology, but in our case at hand the situation is actually quite simple as every element in this sequence will die after at most two steps:

Lemma 2.12. *For any simplicial complex S and $p \in \mathbb{Z}$, the composition $\partial^2\colon C_{p+1}(S) \to C_{p-1}(S)$ of the boundary map with itself is the zero map.*

Proof. We may assume that $p > 0$, since otherwise $C_{p-1}(S) = 0$. As in Remark 2.6, the statement now depends crucially on the sign $(-1)^k$ in Definition 2.5: For any oriented $(p+1)$-simplex

$[i_0,\dots,i_{p+1}]$ we have

$$\begin{aligned}
\partial^2[i_0,\dots,i_{p+1}] &= \partial \sum_{k=0}^{p+1} (-1)^k [\dots,\widehat{i_k},\dots] \\
&= \sum_{k,l:\,l<k} (-1)^l(-1)^k[\dots,\widehat{i_l},\dots,\widehat{i_k},\dots] + \sum_{k,l:\,l>k} (-1)^{l-1}(-1)^k[\dots,\widehat{i_k},\dots,\widehat{i_l},\dots] \qquad (1) \\
&= 0, \qquad (2)
\end{aligned}$$

where the exponent $l-1$ in (1) comes from the fact that the entry i_l is actually at position $i_l - 1$ as i_k is already left out, and (2) follows since the first sum in (1) is just the negative of the second due to the symmetry $k \leftrightarrow l$. □

In fact, this situation of a sequence of vector spaces in which already the composition of any two of its linear maps is zero is so ubiquitous in mathematics that it has a special name (borrowed from algebraic topology).

Definition 2.13 (Chain complexes)**.**

(a) A sequence (C,∂) of vector spaces is called a **chain complex** if $\partial_p \circ \partial_{p+1} = 0$ for all $p \in \mathbb{Z}$, i. e. if $\operatorname{im}\partial_{p+1} \subset \ker \partial_p$.

(b) Morphisms of chain complexes are exactly morphisms as sequences as in Definition 2.8 (b). This makes chain complexes into a category, which we denote by Ch.

(c) We carry over the notations from Definition 2.2 and 2.5: The elements of C_p for $p \in \mathbb{Z}$ will be called ***p*-chains**, and the maps ∂_p are referred to as **boundary maps**. Chains in the kernel of ∂ are called **closed**, or **cycles**, and chains in the image of ∂ are called **boundaries**.

Example 2.14. Lemma 2.12 says precisely that the simplicial chain functor of Construction 2.9 is actually a functor from Simp to Ch. We will therefore call $C(S)$ the **simplicial chain complex** of a
03 simplicial complex S.

Now in the setting of chain complexes, the defining property $\operatorname{im}\partial_{p+1} \subset \ker\partial_p$ means that boundaries are always cycles, so it makes sense to apply our original idea to consider the quotient spaces of cycles modulo boundaries. In fact, almost all homology theories (and clearly all homology theories that we will see in these notes) are obtained by taking cycles modulo boundaries in a suitable chain complex, so let us define homology now in this general situation.

Definition 2.15 (Homology of a chain complex)**.** For a chain complex (C,∂) and $p \in \mathbb{Z}$, its p-th **homology group** is the quotient vector space

$$H_p(C) := \ker\partial_p / \operatorname{im}\partial_{p+1}.$$

Its dimension $b_p(C) := \dim H_p(C) \in \mathbb{N} \cup \{\infty\}$ is called the p-th **Betti number** of C.

Notation 2.16. By construction, the p-th homology group $H_p(C)$ of a chain complex C is a so-called *subquotient* of C_p, i. e. a quotient space of a subspace of C_p. To keep the notations simple, we will not indicate the equivalence classes of this quotient space in any way, i. e. for a p-chain $c \in C_p$ we will just write $c \in H_p(C)$ to indicate that it is closed and determined only up to boundaries.

Lemma 2.17 (Homology as a functor)**.** *For any $p \in \mathbb{Z}$, the p-th homology sending a chain complex C to $H_p(C)$ is a functor from Ch to Vect. In other words, for any morphism $f\colon C \to D$ of chain complexes there is a well-defined induced linear map $H_p(C) \to H_p(D)$, $c \mapsto f(c)$ that is compatible with compositions and the identity. (As in Notation 2.16, we will usually denote it by the same letter f.)*

Proof. As in Definition 2.8, the morphism f gives us a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{p+1} \arrow[r, "\partial"] \arrow[d, "f"] & C_p \arrow[r, "\partial"] \arrow[d, "f"] & C_{p-1} \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & D_{p+1} \arrow[r, "\partial"] & D_p \arrow[r, "\partial"] & D_{p-1} \arrow[r] & \cdots .
\end{tikzcd}$$

Now let a homology class in $H_p(C)$ be given by $c \in C_p$, i. e. we have $\partial c = 0$. By commutativity of the right square above, we then have $\partial f(c) = f(\partial c) = f(0) = 0$, i. e. $f(c) \in D_p$ is a cycle as well. To check well-definedness, let now c be a boundary, i. e. $c = \partial c'$ for some $c' \in C_{p+1}$. Then, by commutativity of the left square we have $f(c) = f(\partial c') = \partial f(c')$, i. e. $f(c)$ is a boundary as well. Hence, the assignment $c \mapsto f(c)$ gives a well-defined linear map from $H_p(C)$ to $H_p(D)$. It is obvious that it is compatible with compositions and the identity. □

As you can see from the previous proof, checking functoriality is often just a straightforward computation. In these cases, we will usually omit such calculations in the future.

But let us come back to topology. Of course, the idea is now to take the homology of the simplicial chain complex of a simplicial complex:

Definition 2.18 (Simplicial homology)**.** For a simplicial complex S and $p \in \mathbb{Z}$, the p-th **simplicial homology group** is defined to be the vector space $H_p(S) := H_p(C(S))$, i. e. the p-th homology group of the simplicial chain complex of S. If we want to indicate the chosen ground field K in the notation, we will write it as $H_p(S,K) = H_p(C(S,K))$. The corresponding Betti numbers are called the **Betti numbers** of S and denoted by

$$b_p(S) := b_p(S,K) = \dim H_p(S,K) \quad \in \mathbb{N} \cup \{\infty\}.$$

Notation 2.19 (Pushforwards for simplicial homology)**.** As the composition of the simplicial chain functor with the homology functor, assigning to a simplicial complex S its p-th homology group $H_p(S)$ for a given $p \in \mathbb{Z}$ is a functor from Simp to Vect. This means that for any morphism $f\colon S \to T$ of simplicial complexes there is an induced morphism $H_p(S) \to H_p(T)$ compatible with compositions and the identity, which is given by sending the class of a p-cycle $c \in C_p(S)$ to the class of the pushforward $f_*(c) \in C_p(T)$ as in Construction 2.9. In accordance with Notation 2.16 and Lemma 2.17, we will write this morphism also just as $f_*\colon H_p(S) \to H_p(T)$ and still call it the *pushforward* by f.

Remark 2.20.

(a) Of course, for any simplicial complex S we have $b_p(S) = 0$ for $p < 0$ or $p > \dim S$: In this case, we already have $C_p(S) = 0$, which implies that $H_p(S) = 0$.

(b) As we have already said in the introduction to this chapter, we will see later that the simplicial homology groups of two simplicial complexes with homeomorphic topological realizations are naturally isomorphic, so we have in fact just constructed vector spaces that see the topological (and not the combinatorial) properties of a simplicial complex. It is almost impossible however to prove this statement directly from our definitions – in fact, we will construct a completely new homology theory called *singular homology* in Chapter 6 which from the very start only needs a topological space as its input, and for which we can then show that it agrees with simplicial homology for topological realizations of simplicial complexes (see Proposition 8.1).

(c) Note that all our constructions to arrive at the simplicial homology groups are phrased in terms of finite-dimensional vector spaces and linear maps between them, all of which are explicitly given by Definition 2.2 and 2.5. Hence, the simplicial homology of a simplicial complex is readily computable using just standard algorithms of linear algebra. In fact, this is the main advantage of simplicial homology – all other homology theories that we will see later on work with infinite-dimensional vector spaces and thus lack this immediate computability. Moreover, the matrices describing the boundary maps in Definition 2.5 are actually very simple: Most of its entries are 0, and all others are ± 1. However, for simplicial complexes occurring in practice the matrices can easily become huge: already for the torus as in Example 1.17 (d) we had 27 one-dimensional simplices, so we would have to deal with vector spaces of this dimension. Hence, we still rely on clever techniques to actually perform the computations with reasonable effort. We will develop such techniques now and in Chapters 3 and 4. Let us start with some easy results by showing that simplicial homology splits on connected components and considering simplicial homology in dimension 0.

Lemma 2.21 (Simplicial homology of connected components)**.** *For a simplicial complex S with connected components $S_1,\dots,S_n$ we have*

$$H_p(S) \cong H_p(S_1) \times \cdots \times H_p(S_n)$$

for all $p \in \mathbb{Z}$ (with an isomorphism given by the pushforwards $H_p(S_i) \to H_p(S)$ for the inclusion morphisms $S_i \to S$ for all $i = 1,\dots,n$). In particular, this means that $b_p(S) = b_p(S_1) + \cdots + b_p(S_n)$.

Proof. By construction of the simplicial chain groups it is clear that $C_p(S) = C_p(S_1) \times \cdots \times C_p(S_n)$ for all $p \in \mathbb{Z}$. As the boundary maps act on each factor separately, it follows immediately by taking cycles modulo boundaries that $H_p(S) \cong H_p(S_1) \times \cdots \times H_p(S_n)$. □

Lemma 2.22 (Degree of a 0-cycle)**.** *For any non-empty simplicial complex S, there is a well-defined surjective linear map*

$$\deg\colon H_0(S) \to K$$

*determined by sending an oriented 0-simplex $[i]$ to 1 for all $i \in V(S)$. We call $\deg c$ the **degree** of $c \in H_0(S)$ (we can think of it as just counting the points in c with their respective coefficients).*

Proof. There is clearly a unique surjective linear map $\deg\colon C_0(S) \to K$ sending each oriented 0-simplex $[i]$ to 1 for $i \in V(S)$ as these 0-simplices form a basis of $C_0(S)$. To see that it passes to a surjective linear map on $H_0(S)$ note first that all of these 0-simplices are also closed since the degree map $\partial\colon C_0(S) \to C_{-1}(S)$ is trivial. Hence it just remains to prove that deg is the zero map on boundaries. But this is obvious since for every oriented edge $[i_0,i_1]$ in S we have $\deg \partial[i_0,i_1] = \deg([i_1]-[i_0]) = 1-1 = 0$. □

Corollary 2.23 (b_0 for a simplicial complex)**.** *For any simplicial complex S, the Betti number $b_0(S)$ is the number of connected components of S. More precisely, a basis of $H_0(S)$ is given by the 0-simplices $[i]$ where we take one arbitrary vertex i from each connected component of S. In particular, the degree map of Lemma 2.22 is an isomorphism if S is connected.*

Proof. By Lemma 2.21 we may assume that S is connected. Note that $\operatorname{im}\partial_1$ is the vector space generated by the 0-chains $\partial[i_0,i_1] = [i_1]-[i_0]$ for all edges $\{i_0,i_1\} \in S$. This means that two oriented 0-simplices $[i_0]$ and $[i_1]$ in $C_0(S)$ are identified in $H_0(S)$ whenever they are connected by an edge in S. But the connectedness of S means that any two vertices are connected by edges by Lemma 1.8 (b), so the class of any one oriented 0-simplex in S suffices to generate $H_0(S)$. In particular, this means that $b_0(S) \le 1$. As Lemma 2.22 implies that we must also have $b_0(S) \ge 1$, the result follows. □

Another trick to obtain some Betti numbers easily is the Euler characteristic; it is a certain combination of the Betti numbers that can be computed without considering the boundary maps at all.

Definition 2.24 (Euler characteristic)**.**

(a) Let C be a chain complex for which all Betti numbers are finite, and only finitely many of them are non-zero. Then

$$\chi(C) := \sum_{p\in\mathbb{Z}} (-1)^p b_p(C)$$

is called the **Euler characteristic** of C.

(b) The Euler characteristic of a simplicial complex S is defined to be $\chi(S) := \chi(C(S))$, i.e. the Euler characteristic of the simplicial chain complex of S.

Lemma 2.25. *Let C be a chain complex such that all vector spaces C_p for $p \in \mathbb{Z}$ are finite-dimensional, and almost all of them are zero. Then*

$$\chi(C) = \sum_{p\in\mathbb{Z}} (-1)^p \dim C_p.$$

Proof. This follows immediately from the dimension formulas of linear algebra, noting that the following sums are actually all finite by assumption:

$$\begin{aligned}
\chi(C) &= \sum_{p\in\mathbb{Z}} (-1)^p \dim(\ker\partial_p/\operatorname{im}\partial_{p+1}) \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim\ker\partial_p + \sum_{p\in\mathbb{Z}} (-1)^{p+1} \dim\operatorname{im}\partial_{p+1} \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim\ker\partial_p + \sum_{p\in\mathbb{Z}} (-1)^{p} \dim\operatorname{im}\partial_{p} \qquad \text{(by index shifting)} \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim C_p. \qquad \square
\end{aligned}$$

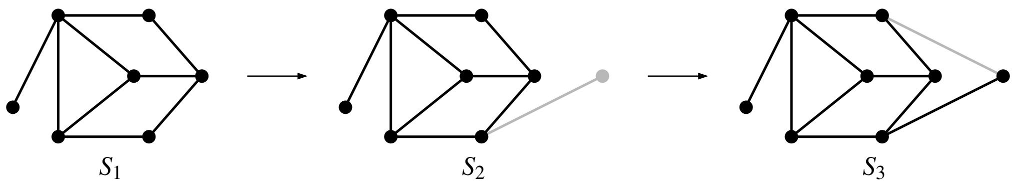
For the remaining part of this chapter, let us now study several examples in which we can already compute the simplicial homology groups with our current methods.

Example 2.26 (Simplicial homology of graphs)**.** Let S be a graph, i. e. a simplicial complex with $\dim S \le 1$. Lemma 2.21 tells us that it is sufficient to consider the case when S is connected, hence $b_0(S) = 1$ by Corollary 2.23. By Remark 2.20 (a) the only other Betti number that is not trivially zero is then $b_1(S)$, and we can compute it using the Euler characteristic as

$$b_1(S) = b_0(S) - (b_0(S) - b_1(S)) = 1 - \chi(S) \overset{2.25}{=} 1 - \dim C_0(S) + \dim C_1(S),$$

i. e. as 1 minus the number of vertices plus the number of edges of S. This number $b_1(S)$ is usually called the *genus* of the graph S. Let us study its geometric interpretation by inductively building up the graph, adding one edge at a time. To do this, we start with a graph S having only one vertex and no edges, which has genus $b_1(S) = 1-1+0 = 0$. Now there are two types of adding an edge, shown in the picture below where we add the gray edge to a given graph:

- We can add an edge that is connected to the previous graph at only one vertex, as for passing from S_1 to S_2 in the picture. This adds one 0-simplex and one 1-simplex to the graph, hence keeps the genus constant. Topologically, we just add an end to the graph somewhere, not affecting any holes in it.
- We can add an edge that is connected to the previous graph at both vertices, as for the transition from S_2 to S_3 in the picture. This only adds one 1-simplex, and thus raises the genus by 1. Geometrically, we close a new circle in the graph in this way. If the graphs are realized by a map $r\colon V(S) \to \mathbb{R}^2$ as in the picture below, we can also interpret this process as raising the number of connected components of $\mathbb{R}^2\backslash r(S)$ by 1.

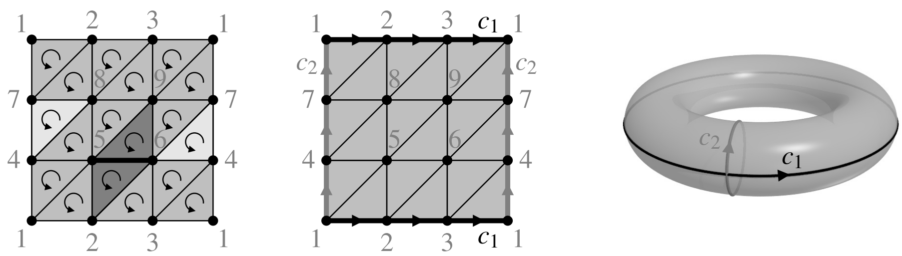


In summary, this means that we can interpret the genus of a graph S as the number of independent circles in the graph, or in the case of a graph realized in $\mathbb{R}^2$ as the number of bounded connected components of $\mathbb{R}^2\backslash r(S)$ (which we can also view as the number of holes in the graph). In this way, we can e. g. readily see without counting vertices and edges that the genus of S_1 and S_2 above is 3, whereas it is 4 for S_3.

Example 2.27 (Simplicial homology of a torus)**.** Let S be the simplicial complex of Example 1.17 (d) realizing a torus, shown again in the picture below on the left. As S is connected, we have $b_0(S) = 1$ again by Corollary 2.23.

Next, we claim that $b_2(S) = 1$ as well. To see this, we first show that any closed 2-chain $c \in C_2(S)$ contains all 2-simplices of S with the same coefficient, taken with the orientation as indicated by the circular arrows. Consider any inner edge of S (i. e. not at the border of the square), such as the thick edge $\{5,6\}$ in the picture. In the boundary $\partial c \in C_1(S)$ it gets a contribution from exactly two 2-simplices, in the example below these are the dark 2-simplices $\{2,5,6\}$ and $\{5,6,9\}$. But $\partial c = 0$, and hence these contributions must cancel. This means that the dark simplices must occur in c with the same coefficient, with the orientation as indicated in the picture. Using this argument for all inner edges, we see that c must in fact contain all 2-simplices with the same coefficient.

Now note that this cancellation of the edges in the boundary ∂c actually works out at the border of the square as well: For example, the edge $\{4,7\}$ (which appears both on the left and right side of the square due to the identification of these sides) gets a contribution in ∂c from the two light-colored 2-simplices $\{4,7,8\}$ and $\{4,6,7\}$. As we already know that they have the same coefficient in c, and the orientations work out as well (the boundary of $[8,7,4]$ contains $[7,4]$, and the boundary of $[4,7,6]$ contains $[4,7] = -[7,4]$), these contributions cancel as well. Hence, we see that the space of closed 2-chains in S is in fact one-dimensional, generated by the cycle containing all 2-simplices of S with coefficient 1 in the orientation below. This 2-cycle is clearly not a boundary since $C_3(S) = 0$, and hence we see that $H_2(S)$ is one-dimensional, i. e. $b_2(S) = 1$.



The computation of $b_1(S)$ is now simple: As the torus has 9 zero-dimensional, 27 one-dimensional and 18 two-dimensional simplices, its Euler characteristic is $\chi(S) = 9 - 27 + 18 = 0$ by Lemma 2.25, and thus we obtain

$$b_1(S) = b_0(S) + b_2(S) - \chi(S) = 1 + 1 - 0 = 2.$$

In fact, we claim that a basis of $H_1(S)$ is given by c_1 and c_2 in the middle picture above (which correspond topologically to the two independent loops in the torus as in the right picture). As c_1 and c_2 are clearly closed and we know already that $\dim H_1(S) = b_1(S) = 2$, it suffices for a proof of this claim to see that these two cycles are linearly independent. Again, with a neat trick we can prove this without huge matrix computations: Let $\lambda_1, \lambda_2 \in K$ with

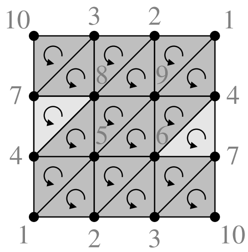
$$\lambda_1 c_1 + \lambda_2 c_2 = 0 \quad \in H_1(S). \tag{$*$}$$

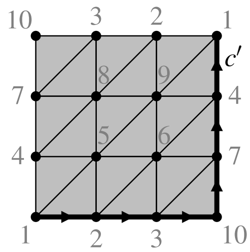
By Example 1.28 (c) there is a morphism $f\colon S \to S^{\{1,2,3\}}$ to the circle $S^{\{1,2,3\}}$ corresponding to the vertical projection in the middle picture above. Its pushforward $f_*\colon H_1(S) \to H_1(S^{\{1,2,3\}})$ as in Notation 2.19 clearly sends c_2 to 0, so applying f_* to $(*)$ yields

$$\lambda_1 f_* c_1 = 0 \quad \in H_1(S^{\{1,2,3\}}).$$

But $f_* c_1 = [1,2] + [2,3] + [3,1]$ is a generator of the 1-dimensional homology group $H_1(S^{\{1,2,3\}})$ by Example 2.26, so we conclude that $\lambda_1 = 0$. Using the same argument for the horizontal projection gives $\lambda_2 = 0$ as well, thus rigorously proving without any matrix computations that c_1 and c_2 are a basis of $H_1(S)$ – as expected from topology.

Example 2.28 (Simplicial homology of the real projective plane)**.** Let us perform the same analysis as in Example 2.27 for the real projective plane S of Example 1.17 (d), shown again in the picture below on the left. It might seem at first that everything works very much in the same way as before, but there is in fact an unexpected twist.





As before, we have $b_0(S) = 1$ since S is connected. It is also still true by the same argument as in Example 2.27 that any 2-cycle in $C_2(S)$ must contain all 2-simplices of S with the same coefficient, in the orientation as in the picture. However, as opposite sides of the square are now identified in opposite directions, the cancellation for the outer edges does not work out any more: For example, the edge $\{4,7\}$ gets a contribution from the light-colored 2-simplices $\{4,7,8\}$ and $\{4,6,7\}$, but this time both contributions from $[8,7,4]$ and $[7,4,6]$ are $[7,4]$, so they get multiplied by 2 instead of canceling out. In other words, if $c \in C_2(S)$ denotes the 2-chain of all 2-simplices with multiplicity 1 then ∂c is now $2c'$ as in the picture above on the right, instead of being zero.

Actually, what this means now *depends on the characteristic of the ground field*:

- If the characteristic is not 2, i. e. if $2 \neq 0$ in K such as e. g. in the case $K = \mathbb{R}$, then c is not a cycle. There are thus no non-zero cycles in $C_2(S)$ at all, i. e. we have $b_2(S,K) = 0$. The Euler characteristic of S is $\chi(S) = 10 - 27 + 18 = 1$, so we obtain
$$b_1(S,K) = b_0(S,K) + b_2(S,K) - \chi(S) = 1 + 0 - 1 = 0.$$
- If the characteristic is 2, i. e. if $2 = 0$ in K such as e. g. in the case $K = \mathbb{Z}_2$, then $\partial c = 2c' = 0$, and thus we obtain $b_2(S,K) = 1$ as in Example 2.27. The Euler characteristic is still the same (in fact, by Lemma 2.25 it is always independent of the ground field), so now we have
$$b_1(S,K) = b_0(S,K) + b_2(S,K) - \chi(S) = 1 + 1 - 1 = 1.$$

Intuitively, the reason for this is that the real projective plane – similarly to the Möbius strip – is *not orientable*. This means that there is no way orient the 2-simplices of S coherently as we have seen above, usually leading to $b_2(S,K) = 0$. However, considering a field of characteristic 2 with $-1 = 1$ essentially means that we ignore orientations (note that Definition 2.2 then identifies all permutations of a simplex), so that then the real projective plane just looks like a closed surface with $b_2(S,K) = 1$.

Exercise 2.29 (Simplicial homology of the Klein bottle)**.** Similarly to Example 1.17 (d), the *Klein bottle* is obtained as a topological space by taking the unit square and identifying the top with the bottom side in the same direction, and the left with the right side in opposite directions.

Starting from a subdivision of the unit square into 18 triangles as in this example, construct a simplicial complex realizing the Klein bottle, and compute its Betti numbers. Do they depend on the chosen ground field?

Exercise 2.30.

(a) Let C be a complex of finite-dimensional vector spaces, with only finitely many C_p non-zero, and let $f: C \to C$ be a morphism. Show that
$$\sum_{p \in \mathbb{Z}} (-1)^p \operatorname{tr}(f: C_p \to C_p) = \sum_{p \in \mathbb{Z}} (-1)^p \operatorname{tr}(f: H_p(C) \to H_p(C)), \qquad (*)$$
where tr denotes the trace of a linear map. If f is the pushforward $C(S) \to C(S)$ by a morphism $g: S \to S$ of simplicial complexes, we call $(*)$ the *Lefschetz number* of g.

(b) Show that any morphism $g: S \to S$ of simplicial complexes with non-zero Lefschetz number has a fixed simplex.

(c) Now suppose that S is a connected simplicial complex with Betti numbers $b_p(S) = 0$ for all $p > 0$. Show that any simplicial morphism $g\colon S \to S$ must fix at least one simplex. Use this to show that any simplicial morphism from a tree (i. e. a connected graph of genus 0) to itself must have a fixed simplex.

Exercise 2.31 (Reduced homology)**.** Sometimes it is more natural to slightly modify our very first Definition 1.2 (a) and drop the condition that a simplex has to be non-empty. The empty set $\emptyset$ is then a valid (-1)-simplex that is a proper face of every other simplex. For a simplicial complex S we can form the *augmented simplicial complex* $\tilde{S} := S \cup \{\emptyset\}$, satisfying again our Definition 1.2 (b) of a simplicial complex with the new notion of faces. As for the constructions in this chapter, every augmented simplicial complex has an oriented (-1)-simplex $[\,]$, and the formula for the boundary map in Definition 2.5 (a) is extended to $p \geq 0$ in the natural way by setting $\partial[i_0] = [\,]$ for all 0-simplices $\{i_0\}$.

If we leave all other constructions of this chapter unchanged, the resulting simplicial homology of the augmented simplicial complex $\tilde{S}$ is called the *reduced simplicial homology* of S and denoted by
04 $\tilde{H}_p(S) := H_p(\tilde{S})$ for all $p \in \mathbb{Z}$. How it is related to $H_p(S)$?