16. Poincaré Duality

In the previous chapters, we have constructed the de Rham cohomology of manifolds and saw that it is naturally isomorphic to singular cohomology. To conclude these notes, we now want to show in this final chapter that our results actually also place some interesting restrictions on the possible values of the Betti numbers of compact oriented manifolds. Due to time constraints, we will only sketch the proofs in this chapter. To simplify notation, we will denote the (de Rham or singular) cohomology groups of a manifold X just by $H^p(X)$ for $p \in \mathbb{Z}$ as in our singular treatment, but we will think of its elements as de Rham cohomology classes, i. e. classes of closed differential forms.

Our starting point is the observation that, for any compact oriented n-dimensional manifold X, combining the wedge product of de Rham cohomology classes of Remark 13.16 with the integral of Remark 14.15 we obtain a well-defined linear map

$$D\colon H^p(X) \to (H^{n-p}(X))^\vee,\ \omega \mapsto \int \omega \wedge \cdot\,. \tag{$*$}$$

It is our goal to show that this is in fact an isomorphism. In particular, this implies the probably surprising statement that the Betti numbers in degree p and $n-p$ of such a manifold must be the same! Moreover, applying this isomorphism twice shows that every cohomology group $H^p(X)$ is isomorphic to its double dual, which is only possible if $H^p(X)$ is finite-dimensional.

The idea to prove this isomorphism will again be a Mayer-Vietoris induction as in Lemma 15.6. This induction principle works for oriented manifolds in the same way as for ordinary manifolds, but clearly not for compact ones, since it relies on building up manifolds from open convex subsets of $\mathbb{R}^n$. Hence, we have to extend our isomorphism statement and thus the map D to non-compact manifolds first – however, this does not work out of the box since then the integral in $(*)$ might be divergent, making D ill-defined. The solution to this problem is to let $D(\omega)$ for $\omega \in H^p(X)$ act only on forms with compact support, i. e. to set up a modified version of de Rham cohomology first that works exclusively with differential forms with compact support.

Construction 16.1 (De Rham cohomology with compact support)**.** For any n-dimensional manifold X and $p \in \{0,\dots,n\}$, we denote by $\Omega_c^p(X) \subset \Omega^p(X)$ the vector subspace of all p-forms on X with compact support. Clearly, the exterior derivative maps forms with compact support to forms with compact support, so we obtain a cochain complex $\Omega_c(X)$. Its cohomology groups $H_c^p(X) := H^p(\Omega_c(X))$ are called the **de Rham cohomology groups with compact support**.

Of course, for a compact manifold X we just have $H_c^p(X) = H^p(X)$ since then the support of any differential form is a closed subset of a compact space, and thus itself compact. For non-compact manifolds however, the compact support condition can change the cohomology groups drastically, as the following simple example shows.

Example 16.2. Let $X = \mathbb{R}$; let us compute $H_c^p(X)$ for $p \in \{0,1\}$.

(a) For $p = 0$, a function $f \in \Omega_c^0(\mathbb{R})$ is closed if and only if $df = f'\,dx$ is zero, which requires f to be constant. But a constant function only has compact support if it is zero, hence $H_c^0(\mathbb{R}) = 0$.

(b) For $p = 1$, every 1-form on $\mathbb{R}$ is closed. Moreover, for 1-forms with compact support there is a linear integration map

$$\Phi\colon \Omega_c^1(\mathbb{R}) \to \mathbb{R},\ f\,dx \mapsto \int f(x)\,dx$$

whose value is also given by $F(\infty)$ for the unique function $F \in \Omega^0(\mathbb{R})$ with $F' = f$ and $F(-\infty) = 0$. Now $f\,dx$ is exact in $\Omega_c(\mathbb{R})$ if and only if F has compact support as well, i. e. if $\Phi(f\,dx) = F(\infty) = 0$. Hence, the cohomology group $H_c^1(\mathbb{R})$ is the quotient $\Omega_c^1(\mathbb{R})/\ker\Phi$, which by the homomorphism theorem is $\operatorname{im}\Phi = \mathbb{R}$. In other words, we have $\dim H^1_c(\mathbb{R}) = 1$, with a generator given by any 1-form $f\,dx$ where f is a function with compact support and non-zero integral, e. g. any bump function as in Construction 11.15. We will call this a *bump 1-form*.

In particular, the dimensions of the de Rham cohomology groups of $\mathbb{R}$ with compact support are exactly opposite to the ordinary ones in Example 13.15. Note that this shows another peculiarity of compactly supported de Rham cohomology, namely that it is not homotopy invariant: The real line is homotopy equivalent to a point, but does not have the same compactly supported de Rham cohomology groups.

Remark 16.3. More surprisingly, the compact support condition also changes the functoriality properties of the cohomology groups: For simplicity, let us only consider the case of inclusion maps $f\colon U \to X$ for an open subset U of a manifold X. In contrast to ordinary de Rham cohomology, there is now no longer a pullback (i. e. restriction) $f^*\colon \Omega^p_c(X) \to \Omega^p_c(U)$ since intersecting the compact support of a form on X with U is in general not compact. However, a form with compact support in U can be extended by zero to a form on X as in Construction 11.15, leading to a *pushforward map* $f_*\colon \Omega^p_c(U) \to \Omega^p_c(X)$! One consequence of this is that reverses the order of the cochain complexes in the Mayer-Vietoris sequence of Proposition 15.4: There is now a sequence

$$0 \longrightarrow \Omega_c(U\cap V) \longrightarrow \Omega_c(U)\times\Omega_c(V) \longrightarrow \Omega_c(U\cup V) \longrightarrow 0$$

whose exactness can be shown using partitions of unity similarly to Proposition 15.4 [BT, Proposition 2.7]. Note that the order of the terms in this sequence is the same as for homology, but it is still a sequence of cochain complexes, with the coboundary operator increasing the degree of the forms. Hence, the corresponding long exact cohomology sequence is

$$\cdots \longrightarrow H^p_c(U\cap V) \longrightarrow H^p_c(U)\times H^p_c(V) \longrightarrow H^p_c(U\cup V) \longrightarrow H^{p+1}_c(U\cap V) \longrightarrow \cdots.$$

Proposition 16.4 (Poincaré duality). *For any n-dimensional oriented manifold X, the map*

$$D\colon H^p(X) \to (H^{n-p}_c(X))^\vee,\ \omega \mapsto \int \omega\wedge\cdot$$

is an isomorphism for all $p = 0,\dots,n$.

Sketch proof. The proof works by Mayer-Vietoris induction as in Lemma 15.6, following the three steps.

As for (a), for a non-empty convex open subset $X \subset \mathbb{R}^n$ we know by the Poincaré Lemma of Corollary 15.3 that $H^p(X)$ is zero for $p \neq 0$ and one-dimensional for $p = 0$. Using an adapted integration homotopy, one can prove a similar Poincaré Lemma for compactly supported de Rham cohomology, showing that

$$H^p_c(X) \cong \begin{cases} \mathbb{R} & \text{for } p = n, \\ 0 & \text{otherwise,} \end{cases}$$

where $H^n_c(X)$ is generated by a bump n-form as in Example 16.2. This shows that D is an isomorphism in this case.

The finite induction step (b) follows as usual from a diagram of Mayer-Vietoris sequences. For open subsets U and V of an oriented manifold X, we have a commutative diagram with exact rows

$$\begin{tikzcd}
H^{p-1}(U)\times H^{p-1}(V) \arrow[r] \arrow[d] & H^{p-1}(U\cap V) \arrow[r] \arrow[d] & H^p(U\cup V) \arrow[r] \arrow[d] & H^p(U)\times H^p(V) \arrow[r] \arrow[d] & H^p(U\cap V) \arrow[d] \\
H^{n-p+1}_c(U)^\vee\times H^{n-p+1}_c(V)^\vee \arrow[r] & H^{n-p+1}_c(U\cap V)^\vee \arrow[r] & H^{n-p}_c(U\cup V)^\vee \arrow[r] & H^{n-p}_c(U)^\vee\times H^{n-p}_c(V)^\vee \arrow[r] & H^{n-p}_c(U\cap V)^\vee,
\end{tikzcd}$$

where the top row is Proposition 15.4, the bottom row is the dual sequence of Remark 16.3, and the vertical maps are given by D.

The infinite induction step (c) follows since disjoint unions of manifolds lead to products of cohomology groups $H^p(X)$ by Remark 15.7 (c), whereas they yield direct sums of compactly supported cohomology groups $H_c^{n-p}(X)$ as in Remark 15.7 (a) (since only finite unions of compact spaces are compact), which are then transformed into products when dualizing by Example 5.13. □

Corollary 16.5. *Let X be an n-dimensional compact oriented manifold, and let $p \in \mathbb{Z}$.*

(a) *The Poincaré duality map gives an isomorphism $H^p(X) \cong (H^{n-p}(X))^\vee$. In particular, we have $b_p(X) = b_{n-p}(X)$.*

(b) *All Betti numbers $b_p(X)$ are finite.*

(c) *For singular homology there are natural isomorphisms $H_p(X) \cong (H^p(X))^\vee \cong H^{n-p}(X)$.*

Proof.

(a) This follows directly from Proposition 16.4 since $H^{n-p}(X) = H_c^{n-p}(X)$ for a compact manifold X.

(b) By Mayer-Vietoris induction it follows that the compactly supported de Rham cohomology of any manifold has at most countable dimension. Now if the dimension of $H^p(X)$ was countably infinite, it would follow by (a) that $H^{n-p}(X) \cong (H^p(X))^\vee$ has uncountable dimension by Example 5.13, which is a contradiction. Hence, $H^p(X)$ must be finite-dimensional.

(c) By Construction 8.10 we have $H^p(X) \cong (H_p(X))^\vee$. Taking duals, we obtain the first isomorphism by (b) since the double dual of a finite-dimensional vector space is isomorphic to the original space. The second isomorphism is just (a). □

Example 16.6.

(a) If X is an n-dimensional connected oriented manifold, the statement $H^0(X) \cong \mathbb{R}$ of Example 13.15 (b) translates into $H_c^n(X) \cong \mathbb{R}$. In particular, if X is also compact then $H^n(X) \cong \mathbb{R}$, i. e. $b_n(X) = 1$.

(b) The sphere S^n is a compact oriented manifold for all n by Example 14.8 (d), and in fact its Betti numbers (which are $b_0(S^n) = b_n(S^n) = 1$ and zero otherwise by Example 8.2 (b)) satisfy the symmetry property of Corollary 16.5 (a). Similarly, this holds for the complex projective spaces $\mathbb{P}^n_\mathbb{C}$ in Example 8.4.

(c) In contrast, the real projective plane $\mathbb{P}^2_\mathbb{R}$ has $b_0(\mathbb{P}^2_\mathbb{R}) = 1 \neq 0 = b_2(\mathbb{P}^2_\mathbb{R})$ over the ground field $\mathbb{R}$ by Example 8.2 (d). As $\mathbb{P}^2_\mathbb{R}$ is a compact 2-dimensional manifold, Corollary 16.5 (a) proves that $\mathbb{P}^2_\mathbb{R}$ is not orientable.

Remark 16.7.

(a) In fact, Poincaré duality can also be established purely in the world of singular homology and cohomology, without referring to de Rham cohomology or differential forms – and thus over any ground field. The general idea to construct the duality map by using a product in cohomology and integrating over an orientation on a manifold is still the same as in our setting, but one needs different (and less intuitive) replacement constructions for this product and the orientations. This generalized version of Poincaré duality does not need smooth structures, but still (topological) manifolds; just topological spaces do not suffice. For details, see e. g. [H, Theorem 3.30].

(b) On the other hand, Corollary 16.5 (c) would allow us to construct *de Rham homology* groups of a compact oriented manifold X e. g. as the duals of $H^p_{\mathrm{dR}}(X)$, so that they are isomorphic to singular homology, but only the analytic methods of de Rham theory are needed. Thinking of $H_p(X)$ in this way, one method to produce an element of it is integration over submanifolds: Given a p-dimensional compact oriented submanifold Y of X, there is a linear map

$$\int_Y : H^p(X) \to \mathbb{R},\ \omega \mapsto \int_Y \omega,$$

defining an element of $(H^p(X))^\vee$, and thus of $H_p(X)$. Hence, such submanifolds determine elements in homology, although homology is not defined in terms of them.

Combining this with Poincaré duality, a submanifold even defines a cohomology class in the following sense.

Construction 16.8 (Poincaré dual of a submanifold)**.** Let X be an n-dimensional compact oriented manifold, and let $Y \subset X$ be a p-dimensional compact oriented submanifold. Then integration over Y defines an element of $(H^p(X))^\vee$ as in Remark 16.7 (b), and thus by Poincaré duality as in Corollary 16.5 (a) an element η_Y of $H^{n-p}(X)$. By the definition of the Poincaré duality map, it is determined by the equation

$$\int_Y \omega = \int_X \eta_Y \wedge \omega \quad \text{for all } \omega \in H^p(X)$$

(i. e. for any closed p-form ω on X). We call η_Y the **Poincaré dual** of Y.

Example 16.9 (Poincaré dual of a point in S^1)**.** Let Y be a point in $X = S^1$. Then by definition the Poincaré dual $\eta_Y \in H^1(S^1)$ satisfies

$$\int_Y f = \int_{S^1} \eta_Y f \tag{$*$}$$

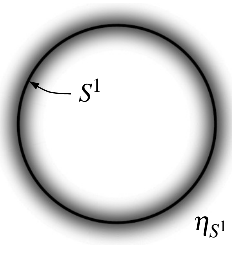
for any closed $f \in \Omega^0(S^1)$. If we forget the closedness condition for a moment this looks as if η_Y is some kind of *Dirac delta function* (resp. functional) [G1, Remark 21.51]: As the left side is just $f(Y)$ and thus does not depend on the values of f at other points, these other values should be zeroed out by η_Y on the right side as well, i. e. η_Y should be zero at all points except Y. On the other hand, for the integrals to match η_Y would have to be infinite at the point Y in such a way that the integral over it is 1 (and thus the integral over $\eta_Y f$ is $1 \cdot f(Y)$).

Of course, such a smooth form η_Y does not exist; it can only be approximated by taking for η_Y a bump 1-form in a small neighborhood of Y with integral 1 as in Example 15.5 (b). Clearly, for arbitrary functions f, the equation $(*)$ would then only be approximately true. But for closed functions $(*)$ becomes an equality again: Such functions must be (locally) constant, hence the right side is just this constant times the integral of η_Y, i. e. again $f(Y)$. This should not really come as a surprise as de Rham's Theorem of Proposition 15.10 tells us that de Rham cohomology classes are topological, i. e. $(*)$ is a topological statement that should not change if we approximate the Dirac delta function by a smooth function.

Remark 16.10 (Localization of Poincaré duals)**.** In fact, our observation of Example 16.9 generalizes: Let Y be a p-dimensional compact oriented submanifold of an n-dimensional compact oriented manifold X. By the Whitney embedding of Proposition 11.21 we may assume that X is embedded in some $\mathbb{R}^m$, and thus that the normal bundle $N_{Y/X}$ is defined as a subbundle of $T_X|_Y$ as in Construction 12.16.

Intuitively speaking, the defining equation of Construction 16.8 says that $\eta_Y \in H^{n-p}(X)$ should be some kind of "Dirac delta form" that is non-zero only on Y, and such that its integral over the $n-p$ normal directions of Y in X is 1. Again, such a form does not exist, but it can be shown that η_Y can be represented by a bump $(n-p)$-form with compact support in an arbitrarily small neighborhood of Y in X [BT, end of §5]. By the Tubular Neighborhood Theorem of 12.21 we may take this neighborhood to be isomorphic to the normal bundle $N_{Y/X}$, and η_Y is then characterized by the condition that its integral over any fiber of this bundle is 1 [BT, Proposition 6.18].

As a concrete visual example, the picture on the right shows a circle $Y = S^1$ in an ambient 2-dimensional compact oriented manifold X that contains a neighborhood of S^1 in $\mathbb{R}^2$. It has a small tubular neighborhood $N_{S^1/X}$ as in Example 12.17 which supports the Poincaré dual $\eta_{S^1} \in H^1(X)$. Concretely, in polar coordinates with radius r we can take $\eta_{S^1} = f(r)\,dr$ where f is a bump function for the radius 1 and with integral 1, so that η_{S^1} integrates to 1 along any radial line intersecting S^1 transversely. In the picture, the value of η_{S^1} resp. f is shown by the shading, with darker colors corresponding to higher values.

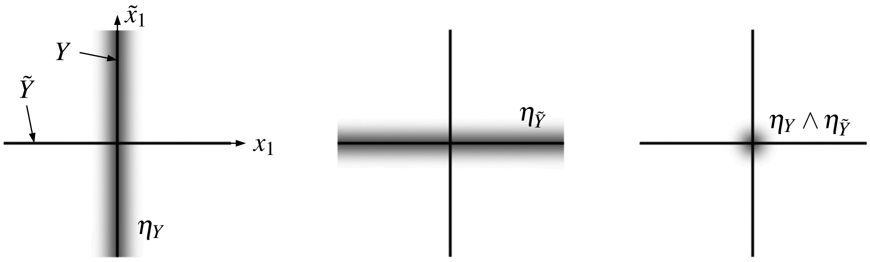


This relation between submanifolds and de Rham cohomology by Poincaré duals can in fact be used to give a geometric interpretation of the ring structure in cohomology, with which we want to end these notes: Under Poincaré duality, taking wedge products of de Rham cohomology classes simply corresponds to intersecting submanifolds.

Proposition 16.11 (Intersection products). *Let Y and $\tilde{Y}$ be compact oriented submanifolds of dimensions p resp. $\tilde{p}$ in an n-dimensional compact oriented manifold X. If Y and $\tilde{Y}$ intersect transversely, i. e. at every intersection point $a \in Y \cap \tilde{Y}$ their normal spaces $N_{Y/X,a}$ and $N_{\tilde{Y}/X,a}$ (considered as subspaces of $T_{X,a}$ as in Remark 16.10) form a direct sum, then $Y \cap \tilde{Y}$ is naturally a compact oriented submanifold of dimension $p + \tilde{p} - n$ in X, and we have*

$$\eta_Y \wedge \eta_{\tilde{Y}} = \eta_{Y \cap \tilde{Y}} \quad \in H^{2n-p-\tilde{p}}(X).$$

Sketch proof. The situation is shown in the picture below: As Y and $\tilde{Y}$ intersect transversely, similarly to Construction 11.2 we can find an adapted chart for X locally around any point $a \in Y \cap \tilde{Y}$ in which Y is given by the vanishing of some coordinates $x_1, \dots, x_{n-p}$, and $\tilde{Y}$ is given by the vanishing of other coordinates $\tilde{x}_1, \dots, \tilde{x}_{n-\tilde{p}}$. By Remark 16.10, the Poincaré duals η_Y and $\eta_{\tilde{Y}}$ can then be localized in small tubular neighborhoods of Y and $\tilde{Y}$, respectively, with integrals 1 along the normal directions at each point. As in the picture below, the wedge product $\eta_Y \wedge \eta_{\tilde{Y}}$ is thus localized in a small neighborhood of $Y \cap \tilde{Y}$, and by Fubini's Theorem its integral over the normal directions of $Y \cap \tilde{Y}$ (which are just the normal directions of Y together with the ones of $\tilde{Y}$) is also 1. Hence, $\eta_Y \wedge \eta_{\tilde{Y}}$ is the Poincaré dual of $Y \cap \tilde{Y}$.



□

Example 16.12 (Bézout's Theorem). As a concluding interesting example of intersection products from the area of algebraic geometry, let us consider the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ of Construction 8.3, which is a compact oriented $2n$-dimensional manifold. Any homogeneous complex polynomial f of degree d in the coordinates $(x_0 : \cdots : x_n)$ of $\mathbb{P}^n_{\mathbb{C}}$ defines a subset

$$Y := \{(x_0 : \cdots : x_n) \in \mathbb{P}^n_{\mathbb{C}} : f(x_0, \dots, x_n) = 0\}$$

which is a submanifold of $\mathbb{P}^n_{\mathbb{C}}$ of dimension $2n-2$ by Proposition 11.9 (a) if f satisfies the corresponding rank condition (which is the case for a sufficiently general choice of f). Expressed over $\mathbb{C}$, this means that Y has complex dimension $n-1$ in the complex n-dimensional projective space; it is called a *hypersurface* in $\mathbb{P}^n_{\mathbb{C}}$. The degree of f is also called the *degree* $\deg Y$ of Y.

Hypersurfaces in projective spaces are among most studied objects in algebraic geometry. One important question about them is the following: Given n hypersurfaces $Y_1, \dots, Y_n$ in $\mathbb{P}^n_{\mathbb{C}}$, a naive dimension count shows that one expects them to intersect in finitely many points – how many intersection points are there? In fact, intersection products can answer this question, by considering the Poincaré duals of various submanifolds of $\mathbb{P}^n_{\mathbb{C}}$. For this, note that any complex p-dimensional submanifold Y of $\mathbb{P}^n_{\mathbb{C}}$ has a natural orientation by Exercise 14.9, so that its Poincaré dual $\eta_Y \in H^{2n-2p}(\mathbb{P}^n_{\mathbb{C}})$ is well-defined, and recall from Example 8.4 that these cohomology groups $H^{2n-2p}(\mathbb{P}^n_{\mathbb{C}})$ are all one-dimensional.

(a) By Example 13.15 (b), any closed 0-form ω on $\mathbb{P}^n_{\mathbb{C}}$ is constant. Hence its integral over any point $P \in \mathbb{P}^n_{\mathbb{C}}$ is the same, which means by Construction 16.8 that all points have the same Poincaré dual $\eta_P \in H^0(\mathbb{P}^n_{\mathbb{C}})$, which is clearly non-zero.

(b) If $Y \subset \mathbb{P}^n_{\mathbb{C}}$ is a hypersurface and $L \subset \mathbb{P}^n_{\mathbb{C}}$ a complex line, then $Y \cap L$ is obtained by solving a polynomial equation of degree $\deg Y$ in one complex variable, leading to $\deg Y$ intersection points if L is general enough so that the intersection is transverse. Hence, by Proposition 16.11 we obtain $\eta_Y \wedge \eta_L = \deg Y \cdot \eta_P$ with $\eta_Y \in H^2(\mathbb{P}^n_{\mathbb{C}})$ and $\eta_L \in H^{2n-2}(\mathbb{P}^n_{\mathbb{C}})$. Note that η_L actually does not depend on the chosen line since this space is one-dimensional and different lines lead to the same result $\det Y \cdot \eta_P$. The same then also holds for Y as long as $\deg Y$ is fixed.

(c) As a special case, a hypersurface of degree 1 (i. e. a linear subspace of $\mathbb{P}^n_{\mathbb{C}}$ of complex dimension $n-1$) is called a *hyperplane*; we denote its Poincaré dual by $\eta_H \in H^2(\mathbb{P}^n_{\mathbb{C}})$. Note that by (b) we have for any hypersurface Y

$$\eta_Y \wedge \eta_L = \deg Y \cdot \eta_P = \deg Y \cdot \eta_H \wedge \eta_L,$$

so as $H^2(\mathbb{P}^n_{\mathbb{C}})$ is one-dimensional we see that $\eta_Y = \deg Y \cdot \eta_H \in H^2(\mathbb{P}^n_{\mathbb{C}})$.

(d) Finally, n (sufficiently general) hyperplanes clearly intersect transversely in one point, i. e. by Proposition 16.11 we have $\eta_H \wedge \cdots \wedge \eta_H = \eta_P$ for the n-fold wedge product of η_H with itself. Hence, we obtain by (c) for n hypersurfaces $Y_1, \ldots, Y_n$

$$\begin{aligned} \eta_{Y_1} \wedge \cdots \wedge \eta_{Y_n} &= \deg Y_1 \cdot \cdots \cdot \deg Y_n \cdot \eta_H \wedge \cdots \wedge \eta_H \\ &= \deg Y_1 \cdot \cdots \cdot \deg Y_n \cdot \eta_P, \end{aligned}$$

so if the hypersurfaces intersect transversely, their number of intersection points is the product $\deg Y_1 \cdot \cdots \cdot \deg Y_n$ of their degrees. This statement is known as *Bézout's Theorem* and arguably the most important intersection-theoretic result in algebraic geometry. 28