15. Comparison of Singular and De Rham Cohomology

In this chapter we now want to prove the long-awaited statement that the analytic de Rham cohomology of a smooth manifold is canonically isomorphic to its singular cohomology, usually called the *de Rham's Theorem* in the literature. The very rough idea of our proof will be a Mayer-Vietoris type induction similarly to Lemma 4.18 in the simplicial case, i. e. we will start with simple base cases and then glue them together to obtain the statement for arbitrary manifolds. In order do to this, we need to address the following three points:

(A) For the induction start similarly to Lemma 4.18 (a), we have to show that the de Rham cohomology of sufficiently simple spaces is trivial.

(B) For the induction step similarly to Lemma 4.18 (b), we need a Mayer-Vietoris statement for de Rham cohomology.

(C) Finally, we have to establish a manifold version of the Mayer-Vietoris induction principle (note that, in contrast to simplicial complexes, a manifold is not built up from finitely many simple pieces in an obvious way).

All three points will require a little bit of work, but fortunately we are well-prepared for it by our previous results. Let us start with (A), which in singular (co-)homology was a homotopy argument in Proposition 3.11 for star-shaped simplicial complexes. We will copy this idea and show that the de Rham cohomology of a star-shaped region $X \subset \mathbb{R}^n$ is trivial, using a homotopy $\Omega^p(X) \to \Omega^{p-1}(X)$ between the identity and a trivial map (in the algebraic sense of Definition 3.1; note that the index now goes down and not up since we are using cohomology with ascending indices). This homotopy will be constructed from integrating a p-form on X along lines from the center point of the star, eating up one of the differentials to leave us with a $(p-1)$-form. In fact, we have seen the very simplest case of this argument already in Example 13.15 (c) where we showed that the first de Rham cohomology of an open interval is trivial since every 1-form on it is the coboundary of its integral.

In order to be able to construct this map, we need a version of integration that only integrates over one of the coordinates, and not over all of them as in Construction 14.11. It will be sufficient for us to do this in $\mathbb{R}^n$, so we do not have to worry about coordinate changes or orientations.

Construction 15.1 (Integration of forms over one variable)**.** Let $U \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with coordinates (x,t) be an open subset for $n \in \mathbb{N}$. We assume that the non-empty fibers of the projection $U \to \mathbb{R}^n$, $(x,t) \mapsto x$ are open intervals of $\mathbb{R}$ containing 0. For all $p \in \mathbb{N}$ we can then construct a linear map $h\colon \Omega^p(U) \to \Omega^{p-1}(U)$ that integrates p-forms over t, i. e. it is defined on generators by

$$h(f\,dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}})(x,t) = \left(\int_0^t f(x,u)\,du \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}}$$

for any form containing dt, and $h(f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = 0$ for any form not containing dt.

There is a projection map $\pi\colon U \to \mathbb{R}^n$, $(x,t) \mapsto x$, and a zero section $s\colon \pi(U) \to U$, $x \mapsto (x,0)$ of π. The composition $s \circ \pi$ then just sets t to 0 on U, but note that this implies on forms that the pullback $(s \circ \pi)^* = \pi^* s^*$ also sets dt to 0.

Lemma 15.2 (Integration homotopy)**.** *In the situation of Construction 15.1, the map h that integrates over t is a homotopy between $\pi^* s^*\colon \Omega^p(U) \to \Omega^p(U)$ and the identity, i. e. we have*

$$d \circ h + h \circ d = \mathrm{id} - \pi^* s^* \quad \text{on } \Omega^p(U)$$

for all $p \in \mathbb{Z}$.

Proof. This is just a straightforward computation, slightly different depending on whether a form contains dt or not: If $\omega = f\,dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}}$ contains dt, setting $F(x,t) := \int_0^t f(x,u)\,du$ and thus $\partial_i F(x,t) = \int_0^t \partial_i f(x,u)\,du$ we have

$$\begin{aligned}
(d \circ h + h \circ d)(\omega) &= \Big(dF + \sum_{i=1}^n h(\partial_i f \underbrace{dx_i \wedge dt}_{=-dt \wedge dx_i}) \Big) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}} \\
&= \Big(f\,dt + \sum_{i=1}^n \partial_i F\,dx_i - \sum_{i=1}^n \partial_i F\,dx_i \Big) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}} \\
&= f\,dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}} \\
&= (\mathrm{id} - \pi^* s^*)(\omega)
\end{aligned}$$

since $\pi^* s^* \omega = 0$ due to the presence of dt in ω. On the other hand, if $\omega = f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ does not contain dt, we clearly have $h(\omega) = 0$, and thus

$$\begin{aligned}
(d \circ h + h \circ d)(\omega) &= h\Big(\frac{\partial f}{\partial t}\,dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \Big) \\
&= (f(x,t) - f(x,0))\,dx_{i_1} \wedge \cdots \wedge dx_{i_p} \\
&= (\mathrm{id} - \pi^* s^*)(\omega).
\end{aligned}$$

□

Corollary 15.3 (Poincaré Lemma). *For every star-shaped open subset $X \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ we have $H^p_{\mathrm{dR}}(X) = 0$ for all $p > 0$.*

Proof. We may assume that the center point of the star is the origin. Let $\omega \in \Omega^p(X)$ be a closed form; we need to show that it is also exact. In order to integrate it along lines starting at the origin, consider the map

$$g \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n,\ (x,t) \mapsto tx$$

on the open subset $U := g^{-1}(X)$, which satisfies the assumptions of Lemma 15.2 since X is star-shaped with center 0. Using the notations of the lemma, we thus obtain for the form $g^*\omega$ on U

$$(d \circ h + h \circ d)(g^*\omega) = g^*\omega - \pi^* s^* g^* \omega \qquad \in \Omega^p(U).$$

In this equation, we first of all have $dg^*\omega = g^* d\omega = 0$ by Lemma 13.7 (b) since ω is closed. Moreover, the composition $g \circ s$ is the constant zero map, hence $\pi^* s^* g^* \omega = 0$ as well by Lemma 13.7 (c) since $p > 0$. We are thus left with

$$(d \circ h)(g^*\omega) = g^*\omega \qquad \in \Omega^p(U).$$

Finally, we pull this equation back along the map $f \colon X \to U$, $x \mapsto (x,1)$ that sets t equal to 1. As $g \circ f = \mathrm{id}$, we obtain by Lemma 13.7 (b) again

$$d f^*(h(g^*\omega)) = \omega.$$

Hence, the form ω is exact, which shows that $H^p_{\mathrm{dR}}(X) = 0$. □

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Next, let us address problem (B), i. e. establish a Mayer-Vietoris statement for de Rham cohomology. In the singular setting in Chapter 7, the main tool to prove Mayer-Vietoris for a union of two open subsets U and V was the construction of a sufficiently fine subdivision of simplices in $U \cup V$ so that each subsimplex lies in U or V, and we had already remarked that the analytic analogue of this is a partition of unity that allows us to write every smooth function on $U \cup V$ as a sum of two smooth functions with support in U and V, respectively. As we have all these tools already at our disposal, the proof of Mayer-Vietoris for de Rham cohomology is in fact quite straightforward.

Proposition 15.4 (De Rham Mayer-Vietoris sequence). *For any two open subsets U and V of a manifold X the sequence of de Rham cochain complexes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(U\cup V) & \longrightarrow & \Omega(U)\times\Omega(V) & \longrightarrow & \Omega(U\cap V) & \longrightarrow & 0 \\ & & \omega & \longmapsto & (\omega|_U,\omega|_V) & & & & \\ & & & & (\omega,\eta) & \longmapsto & \omega|_{U\cap V}-\eta|_{U\cap V} & & \end{array}$$

is exact. In particular, by Remark 5.8 (d) we obtain a functorial long exact sequence in cohomology

$$\cdots \longrightarrow H^p_{\mathrm{dR}}(U\cup V) \longrightarrow H^p_{\mathrm{dR}}(U)\times H^p_{\mathrm{dR}}(V) \longrightarrow H^p_{\mathrm{dR}}(U\cap V) \longrightarrow H^{p+1}_{\mathrm{dR}}(U\cup V) \longrightarrow \cdots.$$

Proof. The only non-trivial part of the statement is that the last map in the sequence is surjective. So let $\omega\in\Omega^p(U\cap V)$ be any p-form on the intersection. Choose a partition of unity (f_U,f_V) subordinate to the open cover (U,V) of $U\cup V$ as in Corollary 11.17. Then we can consider $f_V\,\omega$ to be a smooth form on U which is zero on $U\backslash V$ (and in fact even on the neighborhood $U\backslash\operatorname{supp} f_V$ of any point of $U\backslash V$ in U, which guarantees smoothness). In the same way, $f_U\,\omega$ is a differential form on V, and thus $(f_V\,\omega,-f_U\,\omega)$ is an inverse image of ω under the last map in the sequence since $(f_V\,\omega)|_{U\cap V}-(-f_U\,\omega)|_{U\cap V}=\omega$. □

Example 15.5. As an example of Proposition 15.4 and the general strategy in this chapter, let us compute the de Rham cohomology groups $H^p_{\mathrm{dR}}(S^1)$ of a circle. Clearly, this group is trivial by definition for $p<0$ or $p>1$, and it is isomorphic to $\mathbb{R}$ for $p=0$ by Example 13.15 (b). Hence, it remains to determine $H^1_{\mathrm{dR}}(S^1)$, and from the viewpoint of singular cohomology we clearly expect this Betti number to be 1 as well.

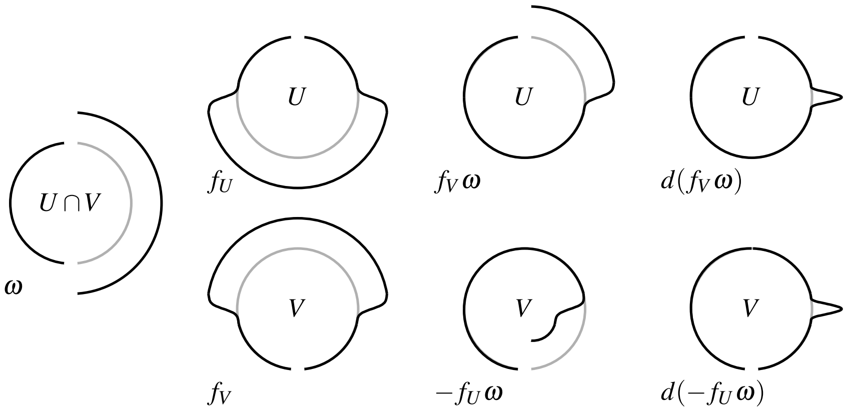
(a) Let us quickly confirm this expectation with with the Mayer-Vietoris sequence analogously to Example 7.11. We write $S^1\subset\mathbb{R}^2$ as the union of two open subsets $U=S^1\backslash\{(0,1)\}$ and $V=S^1\backslash\{(0,-1)\}$ isomorphic to $\mathbb{R}$, with the intersection $U\cap V$ consisting of two disjoint open intervals. By Example 13.15 (b) we then have $b^0_{\mathrm{dR}}(U)=b^0_{\mathrm{dR}}(V)=b^0_{\mathrm{dR}}(U\cup V)=1$ and $b^0_{\mathrm{dR}}(U\cap V)=2$, and by Example 13.15 (c) we have $b^1_{\mathrm{dR}}(U)=b^1_{\mathrm{dR}}(V)=0$. From the exact Mayer-Vietoris sequence of Proposition 15.4

$$0\to\underbrace{H^0_{\mathrm{dR}}(S^1)}_{\cong\mathbb{R}}\to\underbrace{H^0_{\mathrm{dR}}(U)\times H^0_{\mathrm{dR}}(V)}_{\cong\mathbb{R}^2}\to\underbrace{H^0_{\mathrm{dR}}(U\cap V)}_{\cong\mathbb{R}^2}\to H^1_{\mathrm{dR}}(S^1)\to\underbrace{H^1_{\mathrm{dR}}(U)\times H^1_{\mathrm{dR}}(V)}_{=0}$$

we thus conclude by Remark 4.5 (b) that $b^1_{\mathrm{dR}}(S^1)=1$, as expected.

(b) In fact, it is also instructive to explicitly find a generator for $H^1_{\mathrm{dR}}(S^1)$ by following the construction of the connecting map of the long exact cohomology sequence as in the proof of Proposition 4.8. This is shown in the picture below, where the columns from left to right correspond to the steps of the construction, the spaces U, V, and $U\cap V$ are drawn in gray, and the functions resp. forms in black with positive values pointing outwards (where "positive" for the forms in the last column means a positive coefficient function at the differential dx, with x increasing counterclockwise).

More precisely, by (a) we have to start with an element $\omega\in H^0_{\mathrm{dR}}(U\cap V)$ not in the image of $H^0_{\mathrm{dR}}(U)\times H^0_{\mathrm{dR}}(V)$, e. g. with a function which is 1 on one and 0 on the other component of $U\cap V$, as shown in the first column. To construct its image under the connecting map, we follow the proof of Proposition 15.4: To obtain an inverse image of ω in $\Omega^0(U)\times\Omega^0(V)$ we construct a partition of unity (f_U,f_V) subordinate to the cover (U,V) (shown in the second column) and form the pair $(f_V\,\omega,-f_U\,\omega)$ (in the third column). Then we apply the coboundary map d to obtain the pair $(d(f_V\,\omega),d(-f_U\,\omega))\in\Omega^1(U)\times\Omega^1(V)$ (shown in the last column), and by the general construction in the proof of Proposition 4.8 this element must come from a closed form in $\Omega^1(S^1)$ that we are looking for. In fact, one can see in the picture that the two 1-forms $d(f_V\,\omega)$ on U and $d(-f_U\,\omega)$ on V glue to a 1-form on S^1, which is just a bump 1-form giving 1 when integrated over S^1.



(c) Note that this result fits well with our earlier cohomological manifestations of $b_1(S^1) = 1$:

- In simplicial geometry, we have seen in Example 5.11 that a generator of the first cohomology of a simplicial annulus is a cocycle which is almost everywhere 0, except for a small region in which it has values so that it gives 1 on integration over this region. This is precisely the simplicial analogue of the bump 1-form obtained above.
- By Remark 14.15, the integration of 1-forms gives a well-defined map $H^1_{\mathrm{dR}}(S^1) \to \mathbb{R}$, so any 1-form on S^1 (which is necessarily closed) that has a non-zero integral must be a non-trivial de Rham cohomology class, and thus a generator of $H^1_{\mathrm{dR}}(S^1)$. The bump 1-form above clearly satisfies this.
- We had also mentioned already in Example 13.15 (d) that the derivative of the angle in polar coordinates is an example of a closed 1-form on $\mathbb{R}^2\backslash\{0\}$ that is not exact, showing that the first Betti number of this space is non-zero. In fact, this is just another 1-form whose integral around the hole in $\mathbb{R}^2\backslash\{0\}$ is non-zero (namely 2π), and the same holds after restriction to S^1 – so again we see that its integral being non-zero is what makes this cohomology class non-trivial and thus a generator of $H^1_{\mathrm{dR}}(S^1)$.

With our current techniques we could compute and compare the cohomology groups of manifolds if they have a finite open cover such that all possible intersections of the open subsets are isomorphic to star-shaped open subsets of $\mathbb{R}^n$, similarly to the contractible open covers of Definition 8.6. But not all manifolds are of this type – e. g. manifolds with infinitely many connected components, to name a trivial example. Let us therefore now address problem (C) from the introduction to this chapter: In order to establish a Mayer-Vietoris type induction as in Lemma 4.18 for general manifolds we have to include the possibility of infinite unions in some way. Luckily, it turns out that their onion structure ensures that *disjoint* infinite unions suffice, which are then again easy enough to handle.

Lemma 15.6 (Mayer-Vietoris induction for manifolds)**.** *To prove an assertion about all manifolds, it suffices to show:*

(a) *(Induction start) The assertion holds for any manifold isomorphic to a convex open subset of $\mathbb{R}^n$ for any $n \in \mathbb{N}$.*

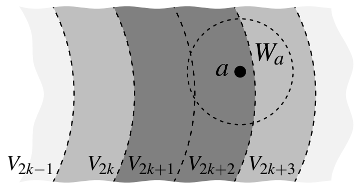
(b) *(Finite induction step) If U and V are open subsets of a manifold X such that the assertion holds for U, V, and $U \cap V$, then it also holds for $U \cup V$.*

(c) *(Infinite induction step) If a manifold X is the disjoint union of countably many submanifolds X_k for $k \in \mathbb{N}$ such that the assertion holds for every X_k, it also holds for X.*

Proof. Starting from convex open subsets of $\mathbb{R}^n$ as in (a), we will show the assertion for progressively larger classes of manifolds in four steps, where steps 1 and 3 use (b), and steps 2 and 4 use (c).

Step 1: The assertion holds for finite unions of convex open subsets of $\mathbb{R}^n$. To see this, write any union $U_1 \cup \cdots \cup U_k \subset \mathbb{R}^n$ of convex open subsets as $U \cup V$ with $U = U_1 \cup \cdots \cup U_{k-1}$ and $V = U_k$. As the intersection of convex subsets of $\mathbb{R}^n$ is convex, the intersection $U \cap V = (U_1 \cap U_k) \cup \cdots \cup (U_{k-1} \cup U_k)$ is a union of $k-1$ convex open subsets of $\mathbb{R}^n$, so the statement follows from (b) by induction on k.

Step 2: The assertion holds for any open subset X of $\mathbb{R}^n$. To prove this, we proceed similarly to the proof of partitions of unity in Proposition 11.16. As X is an n-dimensional manifold, we can pick an onion structure $(V_k)_{k\in\mathbb{N}}$ for X by Proposition 11.13, setting $V_k = \emptyset$ for $k < 0$. For any $k \in \mathbb{N}$ and $a \in \overline{V_{2k+2}} \backslash V_{2k}$ (shown in dark gray in the picture), pick a (convex) open ball W_a around a contained in the open set $V_{2k+3} \backslash \overline{V_{2k-1}}$ (shown in dark and light gray).



As $\overline{V_{2k+2}} \backslash V_{2k}$ is compact, we can pick a finite subset $J \subset \overline{V_{2k+2}} \backslash V_{2k}$ such that $U_k := \bigcup_{a \in J_k} W_a$ covers $\overline{V_{2k+2}} \backslash V_{2k}$. By construction, we then have $X = \bigcup_{k\in\mathbb{N}} U_k$, and every U_k intersects at most the neighboring open subsets U_{k-1} and U_{k+1}.

Now the assertion holds for any U_k by Step 1, and for $U := \bigcup_{k\in\mathbb{N}} U_{2k}$ and $V := \bigcup_{k\in\mathbb{N}} U_{2k+1}$ by (c) since these unions are disjoint. Similarly, it also holds for each intersection $U_k \cap U_{k+1}$ by Step 1, and thus for $U \cap V = \bigcup_{k\in\mathbb{N}} (U_k \cap U_{k+1})$ by (c) as these unions are also disjoint. By (b), the assertion thus holds for $U \cup V = \bigcup_{k\in\mathbb{N}} U_k = X$ as well.

Step 3: The assertion holds for any manifold that can be covered by finitely many charts. To see this, note that the assertion holds for a manifold with a single chart by Step 2, and that the intersection of charts is a chart. Hence, the argument is the same as in Step 1, replacing "convex open subset of $\mathbb{R}^n$" by "chart".

Step 4: The assertion holds for any manifold. Again, this is the same argument as in Step 2, taking for W_a arbitrary charts instead of (convex) open balls, and replacing references to Step 1 by Step 3. □

Remark 15.7. The infinite induction step (c) of Lemma 15.6 is usually easy. Let us quickly check the behavior of our (co-)homology theories for a manifold X that is a disjoint union of countably many components X_n for $n \in \mathbb{N}$:

(a) For singular homology, every singular simplex lies in exactly one component X_n, so every singular chain lies in finitely many components, and the boundary map acts on each component separately. Hence, as in Remark 6.10 (a) the singular homology $H_p(X)$ for $p \in \mathbb{Z}$ is the so-called *direct sum* of all $H_p(X_n)$, i. e. the vector space of all families $(c_n)_{n\in\mathbb{N}}$ with $c_n \in H_p(X_n)$ such that only finitely many c_n are non-zero.

(b) When dualizing, direct sums become ordinary products as in Example 5.13. Hence, the singular cohomology $H^p(X)$ will be the ordinary product of all $H^p(X_n)$ for $n \in \mathbb{N}$ (without any condition that only finitely many components are non-zero).

(c) In de Rham theory, the vector spaces $\Omega^p(X)$ are also visibly the products of $\Omega^p(X_n)$ for all $n \in \mathbb{N}$ since differential forms can be specified on each component separately. The coboundary operator also acts independently on any component, and thus again $H^p_{\mathrm{dR}}(X)$ will be the product of all $H^p_{\mathrm{dR}}(X_n)$ for $n \in \mathbb{N}$.

We are now almost ready to compare de Rham to singular cohomology on manifolds. There is only one piece missing: Recall from Remark 4.20 that to prove the equivalence of two (co-)homology theories using Mayer-Vietoris (and thus the 5-Lemma) we first need a map between the two theories for which we can then prove that it is an isomorphism! As the definitions of de Rham and singular cohomology theories is quite different, this is a bit tricky here – in particular since the singular setting works with *continuous* maps, whereas de Rham cohomology requires *smooth* maps. To deal with this difference, let us construct a smooth version of singular (co-)homology for manifolds.

Construction 15.8 (Smooth singular homology)**.** For a manifold, we can set up a *smooth singular homology theory* in the same way as ordinary singular homology, just requiring the singular simplices to be smooth in the sense of Construction 14.2:

- For $p \in \mathbb{N}$, as in Definition 6.2 a **smooth singular p-simplex** in a manifold X is a smooth map $\sigma\colon |\Delta^p| \to X$. We denote by $C_p^{\mathrm{sm}}(X)$ the K-vector space having them as a basis, and call its elements **smooth singular p-chains** in X. Clearly, $C_p^{\mathrm{sm}}(X)$ is a vector subspace of $C_p(X)$.
- Note that restrictions of smooth maps are smooth, hence the boundary map of Construction 6.5 maps $C_p^{\mathrm{sm}}(X)$ to $C_{p-1}^{\mathrm{sm}}(X)$. We thus obtain a **smooth singular chain complex** $C^{\mathrm{sm}}(X)$ as in Remark 6.7, and corresponding **smooth singular homology groups** $H_p^{\mathrm{sm}}(X) := H_p(C^{\mathrm{sm}}(X))$. Of course, the inclusion $C^{\mathrm{sm}}(X) \to C(X)$ induces natural maps $H_p^{\mathrm{sm}}(X) \to H_p(X)$ between homology groups by Lemma 2.17.

It is now easy to check by the Mayer-Vietoris induction argument of Lemma 15.6 that this natural map is an isomorphism for any manifold X, i. e. that smooth singular homology agrees with ordinary singular homology:

(a) Both singular homologies (smooth and ordinary) are trivial for convex open subsets $X \subset \mathbb{R}^n$: The linear homotopy sending X to any of its points smoothly contracts X to a one-pointed space, so the argument of Example 6.14 (i. e. the proof of Proposition 6.12 and Corollary 6.13) carries over to the smooth case.

(b) The singular Mayer-Vietoris sequence of Corollary 7.10 holds in the same way for smooth singular homology since its main input (namely the subdivision of simplices) does not leave the smooth category. Hence, for any two open subsets U and V of a manifold X we obtain a commutative diagram with exact rows and vertical maps induced by inclusions

$$\begin{array}{ccccccccc}
H_p^{\mathrm{sm}}(U\cap V) & \to & H_p^{\mathrm{sm}}(U)\times H_p^{\mathrm{sm}}(V) & \to & H_p^{\mathrm{sm}}(U\cup V) & \to & H_{p-1}^{\mathrm{sm}}(U\cap V) & \to & H_{p-1}^{\mathrm{sm}}(U)\times H_{p-1}^{\mathrm{sm}}(V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_p(U\cap V) & \longrightarrow & H_p(U)\times H_p(V) & \longrightarrow & H_p(U\cup V) & \to & H_{p-1}(U\cap V) & \to & H_{p-1}(U)\times H_{p-1}(V).
\end{array}$$

By the 5-Lemma 4.19 this means that the smooth singular homology of $U \cup V$ agrees with the ordinary one if this holds for U, V, and $U \cap V$.

(c) For disjoint unions, the statement of Remark 15.7 (a) holds for smooth as well as for ordinary singular homology.

Consequently, by Remark 5.15 (a) singular cohomology can also be defined using smooth simplices, i. e. the singular cohomology groups $H^p(X)$ can also be constructed as the cohomology groups of the cochain complex $(C^{\mathrm{sm}})^\vee(X)$ instead of $C^\vee(X)$.

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Construction 15.9 (Integration over smooth singular chains)**.** We can now construct a natural map from the de Rham cohomology to the singular cohomology of a manifold X: First of all, given a p-form $\omega \in \Omega^p(X)$ and a smooth singular p-simplex $\sigma\colon |\Delta^p| \to X$ as in Construction 15.8, we can form the integral

$$\int_\sigma \omega := \int_{|\Delta^p|} \sigma^*\omega \qquad (*)$$

as in Construction 14.11. Note that this does not need an orientation on X, but only on $|\Delta^p|$ (for which we take the standard one). However, $|\Delta^p|$ is strictly speaking not a manifold with boundary (which would be necessary to apply Construction 14.11), but only a manifold with corners in the sense of Remark 14.5 (d). There are two possible ways to solve this technical problem: Either one sets up all of Chapter 14 for manifolds with corners, or one uses null set arguments to reduce the integral $(*)$ to an integral over the subset of $|\Delta^p|$ on which at most one of the coordinates is zero – which is a manifold with boundary again, with the boundary given by the interiors of the $(p-1)$-dimensional faces of $|\Delta^p|$. Either way, one can make the definition $(*)$ and also the Theorem of Stokes work in this slightly more general situation, and in order to avoid overly technical constructions we will just assume this in what follows.

Extending $(*)$ by linearity, we can define the integral $\int_c \omega$ also for any p-chain $c \in C_p^{\mathrm{sm}}(X)$ with real coefficients, and thus consider integration over ω as an element of $(C_p^{\mathrm{sm}}(X))^\vee$. As this map is linear in ω as well, we obtain a linear integration map

$$\int : \Omega^p(X) \to (C_p^{\mathrm{sm}}(X))^\vee.$$

In fact, by the Theorem of Stokes as in Proposition 14.13 this gives us a morphism of cochain complexes $\Omega(X) \to (C^{\mathrm{sm}})^\vee(X)$, i. e. the diagram

$$\begin{tikzcd} \Omega^p(X) \arrow[r, "d"] \arrow[d, "\int"'] & \Omega^{p+1}(X) \arrow[d, "\int"] \\ (C_p^{\mathrm{sm}}(X))^\vee \arrow[r, "d"] & (C_{p+1}^{\mathrm{sm}}(X))^\vee \end{tikzcd}$$

commutes, where the top row is the exterior derivative and the bottom row the dual of the singular boundary map: Applying first $\int$ and then d to a p-form ω we obtain the linear map $\sigma \mapsto \int_{\partial\sigma} \omega$ in $(C_{p+1}^{\mathrm{sm}}(X))^\vee$, whereas first applying d and then $\int$ gives $\sigma \mapsto \int_\sigma d\omega$.

By Lemma 2.17 and Construction 15.8, the morphism $\Omega(X) \to (C^{\mathrm{sm}})^\vee(X)$ of cochain complexes passes to cohomology to give linear integration maps $H_{\mathrm{dR}}^p(X) \to H^p(X,\mathbb{R})$ for all $p \in \mathbb{Z}$. With all our preparations, it is now in fact easy to prove that this map is an isomorphism.

Proposition 15.10 (De Rham's Theorem). *For any manifold X, the integration over chains map $H_{\mathrm{dR}}^p(X) \to H^p(X)$ of Construction 15.9 is an isomorphism between the de Rham cohomology and the real (smooth) singular cohomology $H^p(X) := H^p(X,\mathbb{R})$ for all $p \in \mathbb{Z}$.*

Proof. We prove the statement by the Mayer-Vietoris induction principle of Lemma 15.6, noting that we already know everything that is necessary:

(a) For an open convex subset of $\mathbb{R}^n$, the de Rham cohomology is trivial by the Poincaré Lemma of Corollary 15.3, and the singular cohomology is trivial by Example 6.14 (and Construction 8.10).

(b) For open subsets U and V of a manifold X, there is a commutative diagram with exact rows

$$\begin{tikzcd}[column sep=small] H_{\mathrm{dR}}^{p-1}(U) \times H_{\mathrm{dR}}^{p-1}(V) \arrow[r] \arrow[d] & H_{\mathrm{dR}}^{p-1}(U\cap V) \arrow[r] \arrow[d] & H_{\mathrm{dR}}^{p}(U\cup V) \arrow[r] \arrow[d] & H_{\mathrm{dR}}^{p}(U) \times H_{\mathrm{dR}}^{p}(V) \arrow[r] \arrow[d] & H_{\mathrm{dR}}^{p}(U\cap V) \arrow[d] \\ H^{p-1}(U) \times H^{p-1}(V) \arrow[r] & H^{p-1}(U\cap V) \arrow[r] & H^{p}(U\cup V) \arrow[r] & H^{p}(U) \times H^{p}(V) \arrow[r] & H^{p}(U\cap V), \end{tikzcd}$$

where the vertical maps are given by Construction 15.9: The top row is the Mayer-Vietoris sequence of Proposition 15.4, the bottom row the Mayer-Vietoris sequence of Construction 8.10 (b), and the diagram commutes since all maps of the corresponding cochain complexes are just given by restrictions. Hence, by the 5-Lemma 4.19, de Rham's Theorem holds for $U \cup V$ if it holds for U, V, and $U \cap V$.

(c) Finally, by Remark 15.7 (b) and (c), de Rham's Theorem holds for countable disjoint unions if it holds for the individual components. □

Example 15.11.

(a) By Proposition 15.10, it follows from Example 8.2 (b) that the only non-trivial de Rham Betti numbers of the sphere S^n with $n \in \mathbb{N}_{>0}$ are $b_{\mathrm{dR}}^0(S^n) = b_{\mathrm{dR}}^n(S^n) = 1$. In particular, for $n = 1$ this reproduces the results of Example 15.5.

(b) Similarly, the only non-trivial de Rham Betti numbers of the punctured plane $X = \mathbb{R}^2\backslash\{0\}$ are $b_{\mathrm{dR}}^0(X) = b_{\mathrm{dR}}^1(X) = 1$, as this space is homotopy equivalent to S^1. Note that we have not shown the homotopy invariance of de Rham cohomology, but it clearly follows now from Proposition 15.10 as we have shown it for singular cohomology in Construction 8.10 (a). One could also prove its homotopy invariance directly by an argument similar to that in the Poincaré Lemma of Corollary 15.3.

In particular, the statement $b_{\mathrm{dR}}^1(X) = 1$ also shows that the non-zero element of $H_{\mathrm{dR}}^1(X)$ that we had found in Example 13.15 (d) is actually a generator.