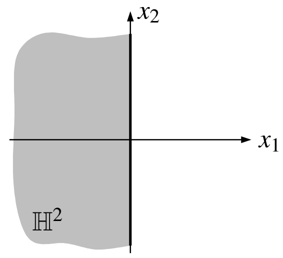
14. Integration of Differential Forms

In the last chapter, we have introduced the concept of the de Rham cohomology groups $H^p(X)$ of a smooth manifold X. Recall that $H^p(X)$ describes differential forms $\omega \in \Omega^p(X)$ that are closed but not exact, i.e. that satisfy $d\omega = 0$ but cannot be written as $\omega = d\eta$ for some $\eta \in \Omega^{p-1}(X)$. Hence, we are facing an "inverse differentiation problem": Given a (closed) p-form ω, is there a $(p-1)$-form η whose exterior derivative is $\omega = d\eta$? From standard analysis, we would clearly expect that such a question is related to integration, so let us study in this chapter how to define and compute integrals of differential forms.

Just a side remark to start with: From standard analysis you would probably also expect that integrals are related to computing volumes under graphs of functions. But manifolds do not carry any kind of natural metric – for example, any two spheres of the same dimension but different radii are isomorphic – so it might seem strange that there is a well-defined notion of integration on such spaces. Actually, it is for exactly that reason why it is not possible to integrate *functions* on a manifold: The integral e.g. over the constant function 1 would have to give the "volume" of the manifold, which is ill-defined. Such integrals over functions can only be constructed for Riemannian manifolds as in Remark 12.29, which do carry a natural metric, but are not in the scope of these notes. However, it turns out that integrals over *differential forms* can be defined on manifolds without referring to a metric. For example, already the standard notation $\int f(x)\,dx$ for one-dimensional integrals over (a subset of) the real line $\mathbb{R}$ suggests that we rather do not integrate a function f but a 1-form $f(x)\,dx$. In fact, a coordinate change to a new variable y with $x = \varphi(y)$ will transform this 1-form into $f(\varphi(y))\,d\varphi(y) = f(\varphi(y))\,\varphi'(y)\,dy$ by Definition 13.3, which then has the same integral as $f(x)\,dx$ due to the integration rule of substitution. For several variables, the transformation rule for multivariate integrals will do the same trick.

Before we can define such integrals on manifolds however, we have to quickly discuss two straightforward extensions of the notion of manifolds. The first one is motivated by the observation that the spaces we want to integrate over are often not quite manifolds due to "boundary points", such as closed intervals in $\mathbb{R}$ as in Example 10.4 (e). To allow this, let us introduce so-called *manifolds with boundary*. Their definition is entirely analogous to that of ordinary manifolds, except that they do not locally look like $\mathbb{R}^n$, but like a closed half-space in $\mathbb{R}^n$ in the sense of the following notation, and as shown in the picture on the right.



Notation 14.1 (Half-spaces)**.** For $n \in \mathbb{N}_{>0}$, we call

$$\mathbb{H}^n := \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_1 \le 0\}$$

the **(standard closed) half-space** in $\mathbb{R}^n$.

Construction 14.2 (Smooth functions on half-spaces)**.** A priori, as in Notation 10.1 smooth maps and diffeomorphisms are only defined in analysis for a map $\varphi\colon U \to V$ between open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ for $m,n \in \mathbb{N}$, in order to be able to define limits for $x \to a \in U$ with x coming from any direction. To be able to apply these notions to half-spaces (which are not of this form) let us define a map $\varphi\colon A \to B$ for arbitrary subsets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ to be smooth if it can be extended to a smooth map from an open neighborhood of A to $\mathbb{R}^m$. As before, we call φ a diffeomorphism if it is bijective, and both φ and φ^{-1} are smooth.

Note that such maps do not have well-defined derivatives in general since the limits needed to construct them may depend on the chosen extensions: For example, if A is a single point in $\mathbb{R}^n$ with $n > 0$ there is clearly no meaningful way to define the derivative of a map $\varphi\colon A \to \mathbb{R}^m$. However, we will compute derivatives of smooth maps in our new sense only on open subsets of $\mathbb{H}^n$, where they can be defined without using extensions at all, just with limits $x \to a \in \mathbb{H}^n$ being one-sided for $x_1 \leq 0$ if $a_1 = 0$.

Construction 14.3 (Manifolds with boundary)**.** Let X be a topological space, and let $n \in \mathbb{N}_{>0}$.

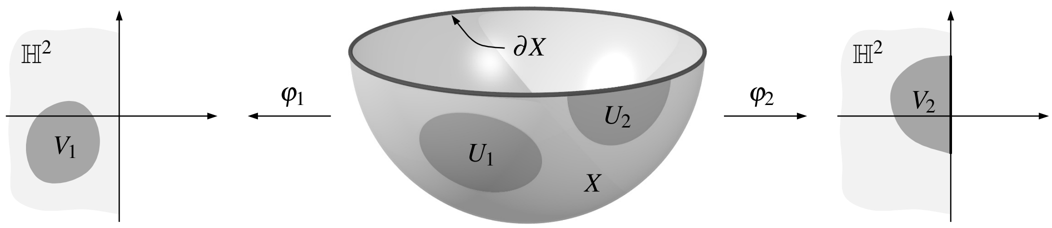
(a) As in Definition 10.2, an (n-dimensional coordinate) *chart* for manifolds with boundary on X is a homeomorphism $\varphi\colon U \to V$ from an open subset $U \subset X$ to an open subset $V \subset \mathbb{H}^n$. Two such coordinate charts $\varphi_1\colon U_1 \to V_1$ and $\varphi_2\colon U_2 \to V_2$ are called *compatible* if their *transition map* $\varphi_2 \circ \varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a diffeomorphism in the sense of Construction 14.2. An (n-dimensional) *atlas* for manifolds with boundary on X is a family of pairwise compatible n-dimensional charts for manifolds with boundary covering X. As in Construction 10.9, an n-dimensional *smooth structure* for a manifold with boundary on X is a maximal atlas for manifolds with boundary on X.

(b) An (n-dimensional) **manifold with boundary** is a second countable Hausdorff space $X \neq \emptyset$ together with an n-dimensional smooth structure in the sense of (a). A morphism of manifolds with boundary is just a smooth map as in Construction 10.5, with smoothness defined as in Construction 14.2.

(c) Note that a transition map $\varphi_2 \circ \varphi_1^{-1}$ as in (a) necessarily sends any point $x \in \mathbb{H}^n$ with $x_1 < 0$ to a point $\varphi_2(\varphi_1^{-1}(x))$ with negative first coordinate as well since $\varphi_2 \circ \varphi_1^{-1}$ is then defined on an open neighborhood of x in $\mathbb{R}^n$, which must be mapped to a neighborhood of $\varphi_2(\varphi_1^{-1}(x)) \subset \mathbb{R}^n$ by the Inverse Function Theorem of Remark 11.4 – but this image neighborhood also has to be contained in $\mathbb{H}^n$, which is only possible if $\varphi_2(\varphi_1^{-1}(x))$ is not on the boundary of $\mathbb{H}^n$.

Hence, there is a well-defined subset $\partial X \subset X$ of all points whose first coordinate in an arbitrary chart is zero; we call it the **boundary** of X. Restricting the charts for X to these points, making them into charts to open subsets of $\mathbb{R}^{n-1}$ by dropping the first coordinate, the boundary ∂X is an $(n-1)$-dimensional manifold (without boundary) contained in X as a closed subset if it is non-empty. Clearly, the complement $X \backslash \partial X$ is an open subset of X that is an n-dimensional manifold without boundary, with charts given by restricting the charts for X to the open subsets with negative first coordinate.

As a visual example, the picture below shows a half-sphere X as a 2-dimensional manifold with boundary; its boundary ∂X is a 1-dimensional circle. We will consider this space rigorously in Example 14.5 (c). Note that a chart $\varphi\colon U \to V$ may or may not touch the boundary ∂X of X, resp. the boundary $\{x \in \mathbb{R}^n : x_1 = 0\}$ of $\mathbb{H}^n$.



Remark 14.4. Most properties of manifolds that we have seen so far carry over to manifolds with boundary in the expected way. For example, submanifolds, vector bundles, and differential forms can be introduced and studied for manifolds with boundary in the same way as for regular manifolds. Most of these extensions are straightforward and only more complicated notationally, and we will not need them for what follows. Hence, let us not go into details here, but only list a few example constructions that will be enough for these notes. The only property of manifolds with boundary that we will actually use is a partition of unity, but it is easy to see that the preparing results of Lemma 11.12 and Proposition 11.13, as well as the final statements of Proposition 11.16 and Corollary 11.17 hold literally in the same way, with (almost) the same proofs.

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Example 14.5.

(a) Clearly, any manifold X with $\dim X > 0$ in the usual sense of Definition 10.16 is also a manifold with boundary, just with $\partial X = \emptyset$.

(b) A general example that covers most use cases is the following: Let $f\colon X \to \mathbb{R}$ be a smooth function on an n-dimensional manifold X with $n > 0$. Assume that f is a submersion, or – to obtain a more general case – that the rank of f is 1 on $f^{-1}(\{0\})$. We then have seen already in Proposition 11.9 (a) and Remark 11.10 (a) that the inverse image $Z := f^{-1}(\{0\})$ is an $(n-1)$-dimensional submanifold of X if it is non-empty. In fact, the very same proof shows that $Y := f^{-1}(\mathbb{R}_{\leq 0}) \subset X$ is an n-dimensional manifold with boundary Z: By the local normal form of Lemma 11.5, the map f is just given by $(x_1,\dots,x_n) \mapsto x_1$ in suitable local coordinates around a point in the boundary, hence restricting these coordinates to $x_1 \leq 0$ will precisely give charts for Y as an n-dimensional manifold with boundary Z. On the other hand, the points not on the boundary just form the open subset $f^{-1}(\mathbb{R}_{<0})$ of X, and thus have charts as well by Example 10.17 (b).

(c) As a concrete example of (b), consider for $n \in \mathbb{N}_{>0}$ the half-sphere

$$Y := \{x \in S^n \subset \mathbb{R}^{n+1} : x_1 \leq 0\}$$

as in the picture of Construction 14.3. By the multivariate version of Example 10.7 (b), the rank of $f\colon S^n \to \mathbb{R}$, $(x_1,\dots,x_{n+1}) \mapsto x_1$ is 1 at all points except $(\pm 1, 0, \dots, 0) \notin f^{-1}(\{0\})$. Hence, we conclude by (b) again that $Y = f^{-1}(\mathbb{R}_{\leq 0})$ is an n-dimensional manifold with boundary $\partial Y = \{x \in S^n : x_1 = 0\} \cong S^{n-1}$.

(d) In the standard smooth structure of $\mathbb{R}^2$, the quadrant $\mathbb{R}^2_{\leq 0}$ is not a manifold with boundary since its boundary $(\mathbb{R}_{\leq 0} \times \{0\}) \cup (\{0\} \times \mathbb{R}_{\leq 0})$ is not a submanifold of $\mathbb{R}^2$. Instead, $\mathbb{R}^2_{\leq 0}$ is a so-called *manifold with corners* – which is defined analogously to a manifold with boundary, except that it does not look locally like $\mathbb{R}^n$ or $\mathbb{H}^n$, but like $\{x \in \mathbb{R}^n : x_i \leq 0 \text{ for all } i\}$.

The second extension of the notion of manifolds that we will need for integration stems from orientation issues. Recall from the introduction to this chapter that the integration rule of substitution [G1, Proposition 12.31], resp. the transformation rule for multivariate integrals [G1, Proposition 30.8] will eventually make our construction of integrals over differential forms coordinate-independent, and thus well-defined on manifolds. However, both these rules are sensitive to a change of orientation, i. e. to space reflections: A one-dimensional integral changes its sign on swapping its lower and upper bound, and the transformation rule for multivariate integrals visibly contains *the absolute value of* the determinant of the Jacobian matrix of the coordinate change in the integral, whereas differential forms are multiplied just with this determinant regardless of its sign by Lemma 13.7 (d). Hence, in order to obtain full coordinate independence we have to remember whether we think of a coordinate chart as containing a space reflection or not, and adjust the sign of the integral accordingly. This leads to the following notation of *oriented manifolds*.

Construction 14.6 (Oriented manifolds)**.** Let $(X, (\varphi_i)_{i\in I})$ be an n-dimensional manifold (with or without boundary), with charts $\varphi_i\colon U_i \to V_i$ for $i \in I$. Recall that the transition maps

$$\varphi_{i,j} := \varphi_j \circ \varphi_i^{-1}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are diffeomorphisms for all $i, j \in I$ by Definition 10.2, so the determinants of their Jacobian matrices are nowhere zero. An **orientation** on X is a collection $(\sigma_i)_{i\in I}$ of smooth (i. e. locally constant) **sign functions** $\sigma_i\colon V_i \to \{\pm 1\}$ such that

$$\sigma_i(x) \cdot \sigma_j(\varphi_{i,j}(x)) \cdot \det(\varphi'_{i,j}(x)) > 0 \quad \text{for all } x \in \varphi_i(U_i \cap U_j),$$

i. e.

$$\sigma_i(x) \cdot \sigma_j(\varphi_{i,j}(x)) \cdot \det(\varphi'_{i,j}(x)) = |\det(\varphi'_{i,j}(x))| \quad \text{for all } x \in \varphi_i(U_i \cap U_j).$$

We say that the manifold is **orientable** if it admits such an orientation; in this case the manifold together with an orientation is called an **oriented manifold**.

Remark 14.7. There are several equivalent definitions of an oriented manifold X, which are all a little more technical than desired. The one of Construction 14.6 will be the most convenient for us, so let us look at its meaning and implications in a bit more detail.

(a) Construction 14.6 corresponds to the intuitive notion of being able to define a consistent orientation on X, where similarly to all our simplicial pictures in Chapter 2 an orientation can be visualized e. g. on a 1-dimensional manifold by specifying a forward or backward direction, and on a 2-dimensional manifold by a circular arrow choosing a clockwise or counterclockwise direction of rotation. Consequently, for example, spheres and the torus are orientable, whereas the Möbius strip and the real projective plane are not (which we will show in Example 16.6 (b)). In the same way as for manifolds with boundary, we do not need to go into details here, but we will consider a few rigorous constructions in Example 14.8 that will be sufficient for our purposes.

(b) It is easy to verify that it suffices to specify sign functions on an arbitrary atlas of X; the condition of Construction 14.6 then ensures that there is always a unique way to extend these functions to all charts of the maximal atlas.

(c) Geometrically, the sign function $\sigma_i\colon V_i \to \{\pm 1\}$ for a chart $\varphi_i\colon U_i \to V_i$ specifies whether the chart transfers the orientation on X to the standard one on $\mathbb{R}^n$ (if the sign is 1) or the opposite one (if the sign is -1). If the chart is connected (which clearly almost all charts are that we use to construct manifolds), this is just a constant, but if U_i and hence also V_i consists of several components it might be that φ_i is orientation-preserving on some components and orientation-reversing on others.

(d) Although the full maximal atlas of an oriented manifold will usually contain charts with both signs, another consequence of (b) is that we can almost always construct oriented manifolds by using charts for which all sign functions are 1: Instead of using a chart with sign function -1 we could equally well just replace one of its coordinates by its negative (i. e. do a space reflection) and use the sign function 1 instead. We can then construct an oriented manifold without specifying any sign functions at all (since they are all 1 anyway), only by Construction 14.6 requiring that all determinants of the Jacobian matrices of the transition maps are positive.

In fact, this would clearly be an easier method of construction resp. definition of oriented manifolds – except that this fails in *exactly two cases*, unfortunately:

- A point (as a 0-dimensional manifold) has *two* orientations, given by the two possible signs ± 1, but this cannot be absorbed into replacing a coordinate by its negative since there is no coordinate.
- Similarly, for a 1-dimensional manifold with boundary such as e. g. the interval $[0,1]$ any chart around the left endpoint would have to map it to the right endpoint 0 of $\mathbb{H}^1$, hence it is necessarily decreasing and cannot reflect the natural orientation of the interval unless we set its sign function to -1.

Example 14.8.

(a) Every open subset $X \subset \mathbb{R}^n$ (or $X \subset \mathbb{H}^n$) is naturally an n-dimensional oriented manifold (with boundary) with the one trivial identity chart and sign function 1, since there are no transition maps to check. We will take this as the standard orientation in this case.

(b) Expanding Example 14.5 (b), let now $f\colon X \to \mathbb{R}$ be a smooth function on an n-dimensional oriented manifold X. Assume again that the rank of f is 1 on $f^{-1}(\{0\})$, so that $Y := f^{-1}(\mathbb{R}_{\leq 0}) \subset X$ is an n-dimensional manifold with boundary. Then Y has a natural orientation as well, obtained by taking only charts which lie in the interior of Y or for which f is in local normal form, and just keeping the sign function for these charts.

(c) Let X be an n-dimensional oriented manifold with boundary; we claim that ∂X has a natural induced orientation. To see this, consider as in Construction 14.3 (c) a chart $\varphi\colon U \to V \subset \mathbb{H}^n$ for X around a point $a \in \partial X$, and make it into a chart $\tilde{\varphi}\colon U \cap \partial X \to \{y \in V : y_1 = 0\} \subset \mathbb{R}^{n-1}$

for ∂X around a by leaving out the first coordinate. The sign function for $\tilde{\varphi}$ will just be the restriction of the sign function for φ.

For $x \in U \cap \partial X$, the transition map between two such charts φ_1 and φ_2 then has Jacobian matrix

$$(\varphi_2 \circ \varphi_1^{-1})'(x) = \left(\begin{array}{c|c} \lambda(x) & 0 \\ \hline * & (\tilde{\varphi}_2 \circ \tilde{\varphi}_1^{-1})'(x) \end{array}\right)$$

for some smooth positive function λ, since it sends points with first coordinate zero to points with first coordinate zero, and points with negative first coordinate to points with negative first coordinate. This means that the determinants of $(\tilde{\varphi}_2 \circ \tilde{\varphi}_1^{-1})'$ and $(\varphi_2 \circ \varphi_1^{-1})'$ have the same sign, and thus that the condition of Construction 14.6 will hold for ∂X if it holds for X, i. e. that ∂X is an oriented manifold as well.

Note however that we have made one (global) choice in this construction: The orientation for $\partial X \subset X$ is determined by first taking a coordinate pointing outwards from ∂X, and then the coordinates for ∂X. We could have made other choices here, but the one above turns out to be most convenient for the theorem of Stokes in Proposition 14.13.

(d) As a concrete example, the closed n-dimensional disk

$$D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\} = f^{-1}(\mathbb{R}_{\leq 0}) \quad \text{with } f\colon \mathbb{R}^n \to \mathbb{R},\ x \mapsto x_1^2 + \cdots + x_n^2 - 1$$

is an n-dimensional manifold with boundary that has a natural orientation by (b) (since the rank of f is only zero at the origin, which is not in $f^{-1}(0)$), and thus its boundary sphere S^{n-1} also has a natural orientation by (c).

Exercise 14.9. We say that a smooth function

$$f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n,\ (x,y) \mapsto (u(x,y), v(x,y))$$

satisfies the *Cauchy-Riemann equations* if

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial v_i}{\partial y_j} \quad \text{and} \quad \frac{\partial u_i}{\partial y_j} = -\frac{\partial v_i}{\partial x_j}$$

for all $i, j = 1, \dots, n$. (One can show that this is equivalent to $f\colon \mathbb{C}^n \to \mathbb{C}^n$ being infinitely complex differentiable if x and y denote the real and imaginary parts of the variables in the source, and u and v denote the real and imaginary parts of the variables in the target.)

Show that any $2n$-dimensional manifold whose transition maps satisfy the Cauchy-Riemann equations (i. e. an n-dimensional complex manifold) is orientable.

Exercise 14.10. Show that the tangent bundle T_X on any manifold X is an orientable manifold.

We are now ready to construct well-defined integrals over n-forms on n-dimensional oriented manifolds (with or without boundary) by transferring them to ordinary integrals on $\mathbb{R}^n$ using coordinate charts. To avoid complications with improper integrals, we will always assume that our differential forms have compact support, so that we only have to integrate smooth and bounded functions. Note that this condition of compact support is automatic if we integrate over a compact manifold, since the support is then always a closed subset of a compact space.

Construction 14.11 (Integration of differential forms with compact support)**.**

(a) Let $\omega \in \Omega^n(V)$ be an n-form with compact support on an open subset V of $\mathbb{R}^n$. Writing it as $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ for a smooth function f with compact support on V, we define its integral by

$$\int_V \omega := \int_V f(x)\, d(x_1, \dots, x_n) \quad \in \mathbb{R}$$

in the usual (Riemann or Lebesgue) sense of multivariate analysis as e. g. in [G1, Definition 29.5]. Note that this is clearly well-defined: As in Construction 11.15, the function f can be extended smoothly by zero to all of $\mathbb{R}^n$, so we can consider its integral over a sufficiently large box containing its support.

(b) The main property of (a) is that it is invariant under signed coordinate transformations in the following sense: For any diffeomorphism $\varphi\colon V \to \tilde{V}$ between open subsets of $\mathbb{R}^n$, sign functions $\sigma\colon V \to \{\pm 1\}$ and $\tilde{\sigma}\colon \tilde{V} \to \{\pm 1\}$ with

$$\sigma(x) \cdot \tilde{\sigma}(\varphi(x)) \cdot \det(\varphi'(x)) = |\det(\varphi'(x))| \quad \text{for all } x \in V \tag{$*$}$$

as in Construction 14.6, and an n-form $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ with compact support on $\tilde{V}$ we have

$$\begin{aligned}
\int_V \sigma \cdot \varphi^* \omega &= \int_V \sigma \cdot \varphi^*(f\, dx_1 \wedge \cdots \wedge dx_n) \\
&\overset{(1)}{=} \int_V \sigma(x) \cdot f(\varphi(x)) \cdot \det \varphi'(x)\, d(x_1, \dots, x_n) \\
&\overset{(*)}{=} \int_V \tilde{\sigma}(\varphi(x)) \cdot f(\varphi(x)) \cdot |\det \varphi'(x)|\, d(x_1, \dots, x_n) \\
&\overset{(2)}{=} \int_{\tilde{V}} \tilde{\sigma}(x) \cdot f(x)\, d(x_1, \dots, x_n) \\
&= \int_{\tilde{V}} \tilde{\sigma} \cdot \omega,
\end{aligned}$$

where (1) follows from Lemma 13.7, and (2) is the transformation rule for multivariate integrals [G1, Proposition 30.8].

(c) Now let $\omega \in \Omega^n(X)$ be an n-form with compact support on an n-dimensional oriented manifold X, and assume for the moment that its support is contained in the open subset $U \subset X$ of a coordinate chart $\varphi\colon U \to V \subset \mathbb{R}^n$ with sign function $\sigma\colon V \to \{\pm 1\}$. We then define its integral as

$$\int_X \omega := \int_V \sigma \cdot (\varphi^{-1})^* \omega,$$

where the integral over V on the right hand side is defined by (a). By our constructions, this is now independent of the chosen chart: If $\tilde{\varphi}\colon \tilde{U} \to \tilde{V}$ is another chart with sign function $\tilde{\sigma}\colon V \to \{\pm 1\}$ and containing the support of ω, we have

$$\int_V \sigma \cdot (\varphi^{-1})^* \omega \overset{13.7\,(c)}{=} \int_V \sigma \cdot (\tilde{\varphi} \circ \varphi^{-1})^* (\tilde{\varphi}^{-1})^* \omega \overset{(b)}{=} \int_{\tilde{V}} \tilde{\sigma} \cdot (\tilde{\varphi}^{-1})^* \omega$$

since the sign functions σ and $\tilde{\sigma}$ satisfy the condition $(*)$ of (b) for the transition map $\tilde{\varphi} \circ \varphi^{-1}$ by Construction 14.6.

(d) Finally, let $\omega \in \Omega^n(X)$ be a general n-form with compact support on an n-dimensional oriented manifold X. By compactness, we can find charts $\varphi_i\colon U_i \to V_i$ with $i \in I$ covering X so that only finitely many of them intersect the support of ω. Choose a partition of unity $(f_i)_{i \in I}$ of X subordinate to this cover in the sense of Corollary 11.17, and set

$$\int_X \omega := \sum_{i \in I} \int_{U_i} f_i \cdot \omega,$$

with the integrals on the right defined by (c), and only finitely many of them being non-zero due to our finiteness assumption. Again, let us check that this is independent of choices: If we pick different charts $\psi_j\colon U'_j \to V'_j$ for $j \in J$ and another partition of unity $(g_j)_{j \in J}$, we obtain

$$\begin{aligned}
\sum_{j \in J} \int_{U'_j} g_j \cdot \omega &= \sum_{i \in I} \sum_{j \in J} \int_{U'_j} f_i g_j \cdot \omega && ((f_i)_{i \in I} \text{ is a partition of unity}) \\
&= \sum_{i \in I} \sum_{j \in J} \int_{U_i \cap U'_j} f_i g_j \cdot \omega && (\text{the support of } f_i g_j \text{ is contained in } U_i \cap U'_j) \\
&= \sum_{i \in J} \int_{U_i} f_i \cdot \omega && (\text{same calculation backwards}).
\end{aligned}$$

Clearly, the same constructions also work for differential forms with compact support on manifolds with boundary, replacing open subsets of $\mathbb{H}^n$ instead of $\mathbb{R}^n$ (and using partitions of unity as in Remark 14.4).

Example 14.12. Construction 14.11 shows rigorously that there is a well-defined notion of integrals over n-dimensional forms with compact support on an n-dimensional manifold, without problems regarding divergent or non-existing integrals. However, it can be shown that for actual computations the formulas above may also be applied without the compact support assumption if one knows beforehand that the integrals actually exist. Let us consider two examples of this in the plane $\mathbb{R}^2$, whose coordinates we will denote by $y = (y_1, y_2)$.

(a) If we want to compute the integral of the 2-form $dy_1 \wedge dy_2$ over the 2-disk $D^2 \subset \mathbb{R}^2$ (with the standard orientation as in Example 14.8 (d)) directly by definition, we would have to take an open cover of the manifold with boundary D^2 by charts whose images lie in $\mathbb{H}^2$, and then use partitions of unity as in Construction 14.11 (d) for this cover. However, it is much easier to use the well-known fact that null sets do not matter for integration, so that we can equally well integrate $dy_1 \wedge dy_2$ over the open 2-disk $\{y \in \mathbb{R}^2 : \|y\| < 1\}$ (although the support of this form is no longer compact), for which we only need the standard chart. As expected, the
25 integral $\int_{D^2} dy_1 \wedge dy_2 = \int_{\|y\|<1} d(y_1, y_2) = \pi$ then just gives the area of the disk.

(b) Let us compute the integral of (the restriction of) the 1-form $y_1\, dy_2$ over the circle $S^1 \subset \mathbb{R}^2$, again with the standard orientation as in Example 14.8 (d). As in (a), rather than covering S^1 with two coordinate charts and using a partition of unity, it is sufficient to consider $S^1 \backslash \{(0,1)\}$, for which we only need one chart. We want to take the chart

$$\varphi_1 : S^1 \backslash \{(0,1)\} \to \mathbb{R},\ (y_1, y_2) \mapsto \frac{y_1}{1 - y_2}$$

of Example 10.4 (b), but have to check first that it has the correct orientation as the boundary of D^2, as otherwise we would get the wrong sign in the result: Locally around e. g. the point $(0, -1) \in S^1$, the map

$$\varphi\colon (y_1, y_2) \mapsto \left(y_1^2 + y_2^2 - 1,\ \frac{y_1}{1 - y_2} \right) \quad \text{with} \quad \det \varphi'(0, -1) = \det \begin{pmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{pmatrix} = 1 > 0$$

gives a chart for D^2 to $\mathbb{H}^2$ with sign function 1, and thus its second component φ_1 is a chart for S^1 also with sign function 1 by Example 14.8 (c).

Finally, we have already computed $(\varphi_1^{-1})^*(y_1\, dy_2) = \frac{8x^2}{(x^2+1)^3}\, dx$ in Example 13.12 (a), and hence we obtain by Construction 14.11 (c)

$$\int_{S^1} y_1\, dy_2 = \int_{\mathbb{R}} \frac{8x^2}{(x^2+1)^3}\, dx = \lim_{c \to \infty} \left[\frac{x(x^2-1)}{(x^2+1)^2} + \arctan x \right]_{-c}^{c} = \pi.$$

In fact, the equality of the two integrals in Example 14.12 is not a coincidence: Note that the form $dy_1 \wedge dy_2$ in (a) is just the exterior derivative of $y_1\, dy_2$ in (b), and the sphere S^1 in (b) is the boundary of the disk D^2 in (a). The following theorem of Stokes – which is the main result of this chapter – asserts that taking exterior derivatives of forms is compatible in general with boundaries of manifolds in this sense. Given the somewhat complicated computations in the example above its proof might seem surprisingly simple, but in fact we have already done the most important part of the work by setting up a coordinate-invariant theory of differentiation and integration. This allows us to reduce the general statement to local statements on $\mathbb{R}^n$ and $\mathbb{H}^n$ which are easy to verify directly.

Proposition 14.13 (**Theorem of Stokes**). *For any $n \in \mathbb{N}_{>0}$ and an $(n-1)$-form ω with compact support on an n-dimensional oriented manifold X with boundary we have*

$$\int_{\partial X} \omega = \int_X d\omega.$$

Proof. Both sides of the equation are linear in ω, so using the coordinate pullbacks and partitions of unity of Construction 14.11 (c) and (d) we may assume that X is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and thus in fact by extension by zero that ω is a form with compact support on $X = \mathbb{R}^n$ or $X = \mathbb{H}^n$ itself. Moreover, linearity implies that we can take the sign function to be 1 and consider a form

$$\omega = f\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \quad \Rightarrow \quad d\omega = (-1)^{i-1}\,\partial_i f\,dx_1 \wedge \cdots \wedge dx_n$$

for a smooth function f with compact support, where as usual $\widehat{dx_i}$ denotes that this differential is left out. Let us now compute the integrals on both sides of the theorem, noting that the order of integration is arbitrary by Fubini's theorem [G1, Chapter 28.B].

If $i > 1$, both sides of the theorem are zero: the integral $\int_{\partial X} \omega$ vanishes since $dx_1 = 0$ on ∂X (which is empty for $\mathbb{R}^n$ and given by $x_1 = 0$ on $\mathbb{H}^n$), and $\int_X d\omega$ is zero since $\int \partial_i f(x)\,dx_i$ gives the difference of f at points with $x_i \to \infty$ and $x_i \to -\infty$, which are both zero since f has compact support.

If $i = 1$ and $X = \mathbb{R}^n$, the same argument holds by symmetry of the coordinates.

Finally, if $i = 1$ and $X = \mathbb{H}^n$ we get

$$\int_X d\omega = \int_{\mathbb{H}^n} \partial_1 f(x_1,\dots,x_n)\,d(x_1,\dots,x_n) \overset{(*)}{=} \int_{\mathbb{R}^{n-1}} f(0,x_2,\dots,x_n)\,d(x_2,\dots,x_n) = \int_{\partial X} \omega,$$

where $(*)$ is obtained by integrating $\partial_1 f$ over x_1 from $-\infty$ (where f is zero due to its compact support) to 0. $\square$

Example 14.14. Note that we had already seen a discrete simplicial version of Proposition 14.13 in Example 5.4 and Remark 5.5. Correspondingly, there are similar examples now that use actual differentiation and integration instead of the simplicial coboundary operator and the pairing between simplicial homology and cohomology:

(a) A closed interval $[a,b] \subset \mathbb{R}$ is a 1-dimensional oriented manifold with boundary $\{a,b\}$, where b has sign 1 and a has sign -1. Applying Proposition 14.13 to a smooth function $f \in \Omega^0([a,b])$ thus gives

$$f(b) - f(a) = \int_{\partial[a,b]} f = \int_{[a,b]} df = \int_a^b f'(x)\,dx,$$

which is just the Fundamental Theorem of Calculus.

(b) Let $X \subset \mathbb{R}^3$ be a 2-dimensional oriented manifold with boundary, and let $\omega \in \Omega^1(X)$ be the restriction of a 1-form on $\mathbb{R}^3$, so that we can write it as $\omega = f_1\,dx_1 + f_2\,dx_2 + f_3 dx_3$ for smooth functions $f_1, f_2, f_3 \colon \mathbb{R}^3 \to \mathbb{R}$. Then the statement $\int_{\partial X} \omega = \int_X d\omega$ translates into

$$\begin{aligned}&\int_{\partial X} (f_1\,dx_1 + f_2\,dx_2 + f_3 dx_3)\\ &\quad = \int_X \big((\partial_1 f_2 - \partial_2 f_1)\,dx_1 \wedge dx_2 + (\partial_2 f_3 - \partial_3 f_2)\,dx_2 \wedge dx_3 + (\partial_3 f_1 - \partial_1 f_3)\,dx_3 \wedge dx_1\big).\end{aligned}$$

As the three functions integrated over X are just the curl of f in the sense of Example 13.5 (c), this statement is usually called the *curl theorem of Stokes*.

(c) Similarly, if $X \subset \mathbb{R}^3$ is a 3-dimensional oriented manifold with boundary and $\omega \in \Omega^2(X)$, we have $\omega = f_1\,dx_2 \wedge dx_3 + f_2\,dx_3 \wedge dx_1 + f_3\,dx_1 \wedge dx_2$ for smooth functions $f_1, f_2, f_3 \colon X \to \mathbb{R}$, and the Theorem of Stokes becomes

$$\int_{\partial X} \big(f_1\,dx_2 \wedge dx_3 + f_2\,dx_3 \wedge dx_1 + f_3\,dx_1 \wedge dx_2\big) = \int_X (\partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3)\,dx_1 \wedge dx_2 \wedge dx_3.$$

This time, the function occurring in the integral over X is the divergence of f as in Example 13.5 (c), and thus this statement is named the *divergence theorem of Gauss*.

Hence, the Theorem of Stokes of Proposition 14.13 unifies several classical theorems from vector calculus, and generalizes them to arbitrary dimension.

Remark 14.15 (Integrals in cohomology)**.** From the topological point of view, the Theorem of Stokes will have many implications in de Rham cohomology since it relates the exterior derivative (i. e. the coboundary operator of the de Rham cochain complex) to the geometric boundary of manifolds – similarly to the corresponding statement in simplicial geometry in Example 5.4. For the moment, let us just note that on a compact n-dimensional oriented manifold X the integration of forms yields a well-defined linear integration map

$$\int : H^n_{\mathrm{dR}}(X) \to \mathbb{R},\ \omega \mapsto \int_X \omega,$$

since X has empty boundary, and thus the integral vanishes on exact forms. In fact, this map will turn out to be an isomorphism if X is connected; we will study this in more detail in Proposition 16.4 and Example 16.6 (a).

Exercise 14.16.

(a) Show that an n-dimensional manifold X is orientable if and only if there is a nowhere-vanishing n-form on X.

(b) Now let X be a compact n-dimensional orientable manifold with boundary. Use (a) to show that ∂X is not a smooth deformation retract of X, and give a counterexample of this statement when X is non-compact.