De Rham Cohomology

13. Differential Forms and De Rham Cohomology

In the last part of these notes, we now want to study algebraic topology for manifolds, using analytic methods. With the idea of singular homology in mind, where elements of the p-th homology group of a topological space can be thought of as being represented by "closed p-dimensional objects" in this space, it might seem that the best starting point for an analytic version of the p-th homology of a manifold X would be to consider compact p-dimensional submanifolds of X. However, we had seen in Chapter 5 already that the more natural approach in analysis is to start with the dual notion of cohomology, in which we study functions on X and the coboundary operator is some kind of differentiation. This is the idea of the so-called *de Rham cohomology* of X that we are going to introduce in this chapter. Note that this does not mean that the idea of analytic homology being represented by compact submanifolds is wrong – actually, we will see in Example 16.9 that certain p-dimensional submanifolds of X do in fact determine "analytic p-dimensional homology classes" in X, it is just not the easiest way to start our journey.

It might look strange at first to try to use a differentiation operator as the coboundary operator for a cohomology theory: After all, we know that such a coboundary operator d must satisfy $d^2 = 0$, whereas second derivatives clearly do not vanish in general. However, we know that higher partial derivatives commute, i. e. that $\partial_i \partial_j f = \partial_j \partial_i f$ for all $i, j \in \{1, \dots, n\}$ and smooth functions $f\colon U \to \mathbb{R}$ on an open subset U of $\mathbb{R}^n$. Hence, if we build some antisymmetry into the theory that allows us to count $\partial_j \partial_i f$ with a different sign than $\partial_i \partial_j f$ in second derivatives (similarly to the factor $(-1)^k$ in Definition 2.5 for the simplicial boundary operator ∂, leading to $\partial^2 = 0$ in Lemma 2.12), we can make the equation $d^2 = 0$ work. This is achieved by the notion of *differential forms*, which we want to introduce first. We will start with their construction on open subsets of $\mathbb{R}^n$, and then carry them over to manifolds using the charts of an atlas.

Construction 13.1 (Differential forms on $\mathbb{R}^n$)**.** Let $U \subset \mathbb{R}^n$ be an open subset with $n \in \mathbb{N}$.

(a) For $p \in \mathbb{N}$ with $p \le n$, a **differential form** of **degree** p or simply **p-form** on U is a family $(f_{i_1,\dots,i_p})_{i_1,\dots,i_p}$ of $\binom{n}{p}$ smooth functions $f_{i_1,\dots,i_p}\colon U \to \mathbb{R}$ with $1 \le i_1 < \cdots < i_p \le n$. We will write it as

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} f_{i_1,\dots,i_p}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

for formal symbols $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. The vector space of all p-forms on U (with component-wise addition and scalar multiplication) will be denoted by $\Omega^p(U)$, where we understand $\Omega^0(U)$ to be the space of smooth functions on U as in Notation 10.1 (a). Moreover, as a notational convention, we set $\Omega^p(U) := \{0\}$ for all $p \in \mathbb{Z}$ with $p < 0$ or $p > n$.

We want to consider the formal expression $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ as alternating in the coordinates, i. e. we set

- $dx_{i_{\pi(1)}} \wedge \cdots \wedge dx_{i_{\pi(p)}} := \operatorname{sign}(\pi) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p} \in \Omega^p(U)$ for all $i_1 < \cdots < i_p$ and permutations π of $\{1, \dots, p\}$, and
- $dx_{i_1} \wedge \cdots \wedge dx_{i_p} := 0 \in \Omega^p(U)$ if $i_1, \dots, i_p$ are not pairwise distinct.

In this way, expressions of the form $f\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \in \Omega^p(U)$ are defined for all smooth functions $f\colon U \to \mathbb{R}$ and $i_1, \dots, i_p \in \{1, \dots, n\}$, and they generate $\Omega^p(U)$ as a real vector space.

(b) As a vector space, note that $\Omega^p(U)$ is just the $\binom{n}{p}$-fold product of $\Omega^0(U)$ with itself. However, differential forms also admit a natural bilinear product called the **wedge product**

$$\Omega^p(U) \times \Omega^q(U) \to \Omega^{p+q}(U),\ (\omega,\eta) \mapsto \omega \wedge \eta$$

for all $q \in \mathbb{N}$ with $q \le n$, defined on generators by

$$(f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \wedge (g\,dx_{j_1} \wedge \cdots \wedge dx_{j_q}) := fg\,dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}$$

for all $f,g \in \Omega^0(U)$ and $i_1,\dots,i_p,j_1,\dots,j_q \in \{1,\dots,n\}$.

It is obvious that this product is well-defined in the sense that it is compatible with the definition of $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ and $dx_{j_1} \wedge \cdots \wedge dx_{j_q}$ in the case of indices that are not strictly increasing.

Remark 13.2.

(a) Clearly, the wedge product of Construction 13.1 (b) is associative. Moreover, the formal symbols $dx_{i_1} \wedge \cdots \wedge dx_{i_p} \in \Omega^p(U)$ of Construction 13.1 (a) can actually be interpreted as the wedge products of $dx_{i_1},\dots,dx_{i_p} \in \Omega^1(U)$.

If you know this construction from commutative algebra, you will probably realize that $\Omega^p(U)$ is the so-called *alternating tensor product* of a free rank-n module $\Omega^1(U)$ over the ring $\Omega^0(U)$, where the basis elements of this module are written as $dx_1,\dots,dx_n$. The wedge product then makes the direct sum of all $\Omega^p(U)$ into an associative algebra over $\Omega^0(U)$, which is called the *exterior algebra* over $\Omega^1(U)$. In what follows, we will not use this terminology however.

(b) The wedge product is commutative up to sign in the following sense: Note that

$$(dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \wedge (dx_{j_1} \wedge \cdots \wedge dx_{j_q}) = (-1)^{pq} (dx_{j_1} \wedge \cdots \wedge dx_{j_q}) \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_p})$$

since we can transform the ordering on the left to the ordering on the right by moving each $dx_{i_p},\dots,dx_{i_1}$ by q positions to the right, leading to pq transpositions and thus to a sign of $(-1)^{pq}$. Hence, we have

$$\omega \wedge \eta = (-1)^{pq}\, \eta \wedge \omega \quad \text{for all } \omega \in \Omega^p(U) \text{ and } \eta \in \Omega^q(U)$$

with $U \subset \mathbb{R}^n$ open.

(c) The formal symbols dx_i for $i = 1,\dots,n$ in Construction 13.1 are usually called *differentials*. They can be interpreted geometrically as an "infinitesimally small change in the variable x_i", similarly to standard analysis when you write a one-dimensional integral as $\int f(x)\,dx$ rather than $\int f$ (in fact we will see in Chapter 14 that well-defined integrals over one-dimensional objects require 1-forms and not functions), or when you just plainly write the derivative of a function $f\colon \mathbb{R} \to \mathbb{R}$ as $f' = \frac{df}{dx}$. In fact, let us now define the derivative of differential forms in this spirit. It is usually called the *exterior derivative* since its values lie in an exterior algebra in the sense of (a).

Definition 13.3 (Exterior derivative)**.** Let $U \subset \mathbb{R}^n$ be an open subset with $n \in \mathbb{N}$.

(a) For a smooth function $f \in \Omega^0(U)$ we define

$$df := \sum_{i=1}^{n} \partial_i f \cdot dx_i \quad \in \Omega^1(U).$$

(b) For a p-form $f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ with $f \in \Omega^0(U)$ we set

$$d(f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}) := df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \quad \in \Omega^{p+1}(U)$$

with df as in (a), and extend this by linearity to a map $d\colon \Omega^p(U) \to \Omega^{p+1}(U)$.

For all $p \in \mathbb{N}$, the map $d\colon \Omega^p(U) \to \Omega^{p+1}(U)$ is called the **(exterior) derivative**.

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Lemma 13.4 (Properties of the exterior derivative)**.** *For any open subset $U \subset \mathbb{R}^n$, $\omega \in \Omega^p(U)$, and $\eta \in \Omega^q(U)$ we have*

(a) $d^2\omega = 0$;

(b) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p\, \omega \wedge d\eta$.

Proof. By linearity, it suffices to check both equations on generators $\omega = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ and $\eta = g\, dx_{j_1} \wedge \cdots \wedge dx_{j_q}$.

(a) We have

$$d^2\omega = d\left(\sum_{i=1}^n \partial_i f \cdot dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}\right) = \sum_{i,j=1}^n \partial_j \partial_i f \cdot dx_j \wedge dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

which vanishes since $\partial_j \partial_i f = \partial_i \partial_j f$ [G1, Proposition 26.6] and $dx_j \wedge dx_i = -dx_i \wedge dx_j$.

(b) By the usual product rule for (multivariate) derivatives [G1, Proposition 25.29 (c)] we obtain

$$\begin{aligned} d(\omega \wedge \eta) &= \left(\sum_{i=1}^n \partial_i f \cdot g + f \cdot \partial_i g\right) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q} \\ &= \sum_{i=1}^n \partial_i f \cdot dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge g\, dx_{j_1} \wedge \cdots \wedge dx_{j_q} \\ &\qquad + (-1)^p \sum_{i=1}^n f\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge \partial_i g \cdot dx_i \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q} \\ &= d\omega \wedge \eta + (-1)^p\, \omega \wedge d\eta, \end{aligned}$$

where the factor $(-1)^p$ arises from shifting dx_i by p positions to the right in the second sum. □

Example 13.5.

(a) Note that the name dx_i for the formal symbols in Construction 13.1 plays well with the notation d for the exterior derivative, since dx_i is actually the exterior derivative of the i-th coordinate function $x \mapsto x_i$ for $i = 1, \dots, n$.

(b) For the smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, $x \mapsto x_1 x_2$ we have $df = x_2\, dx_1 + x_1\, dx_2 \in \Omega^1(\mathbb{R}^2)$, and consequently

$$d^2 f = d(x_2\, dx_1 + x_1\, dx_2) = dx_2 \wedge dx_1 + dx_1 \wedge dx_2 = 0 \quad \in \Omega^2(\mathbb{R}^2)$$

in accordance with Lemma 13.4 (a).

(c) On an open subset U of $\mathbb{R}^3$, the exterior derivative reproduces the standard differential operators of vector analysis:

- For a smooth function $f \in \Omega^0(U)$ we have

$$df = \partial_1 f \cdot dx_1 + \partial_2 f \cdot dx_2 + \partial_3 f \cdot dx_3 \quad \in \Omega^1(U),$$

so written as a triple as in Construction 13.1 (a) this is just $(\partial_1 f, \partial_2 f, \partial_3 f)$, i. e. the *gradient* $\operatorname{grad} f$ of f.

- If we consider a given vector field $f = (f_1, f_2, f_3) : U \to \mathbb{R}^3$ in the same way as a 1-form $f = f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3 \in \Omega^1(U)$, we have

$$df = (\partial_1 f_2 - \partial_2 f_1)\, dx_1 \wedge dx_2 + (\partial_2 f_3 - \partial_3 f_2)\, dx_2 \wedge dx_3 + (\partial_3 f_1 - \partial_1 f_3)\, dx_3 \wedge dx_1$$

in $\Omega^2(U)$. Note that 2-forms in $\mathbb{R}^3$ have three component functions as well. Written again as a triple $df = g_1\, dx_2 \wedge dx_3 + g_2\, dx_3 \wedge dx_1 + g_3\, dx_1 \wedge dx_2$, the vector field

$$g = (\partial_2 f_3 - \partial_3 f_2, \partial_3 f_1 - \partial_1 f_3, \partial_1 f_2 - \partial_2 f_1) : U \to \mathbb{R}^3$$

is called the *curl* of f as in Remark 5.5 (a), denoted by $\operatorname{curl} f$.

- Finally, if we write a given vector field $f = (f_1, f_2, f_3) : U \to \mathbb{R}^3$ as above as 2-form $f = f_1\, dx_2 \wedge dx_3 + f_2\, dx_3 \wedge dx_1 + f_3\, dx_1 \wedge dx_2$, its exterior derivative is

$$df = (\partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3)\, dx_1 \wedge dx_2 \wedge dx_3 \quad \in \Omega^3(U),$$

which is a 3-form whose only component $\partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3$ is called the *divergence* $\operatorname{div} f$ of f as in Remark 5.5 (b).

Note that the equation $d^2\omega = 0$ of Lemma 13.4 (a) thus means that $\operatorname{curl}\operatorname{grad} = 0$ (when applied to 0-forms) and $\operatorname{div}\operatorname{curl} = 0$ (when applied to 1-forms), which are well-known relations in vector analysis.

Of course, this relation $d^2\omega = 0$ for $\omega \in \Omega^p(U)$ with $U \subset \mathbb{R}^n$ would already allow us to construct analytic cohomology groups for such open subsets U of $\mathbb{R}^n$. But let us first transfer our definitions to general manifolds, where differential forms will become sections of suitable vector bundles. We will set up these vector bundles using transition maps as in Construction 12.23, so we need to study how differential forms behave under coordinate transformations. In fact, differential forms will be a contravariant functor, with their pullback being given by the following definition.

Definition 13.6 (Pullback of differential forms)**.** Let $f\colon U \to V$ be a smooth map between open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ with component functions $f_1, \dots, f_m$. Then we define a **pullback** map $f^*\colon \Omega^p(V) \to \Omega^p(U)$ for all p by linear extension of

$$f^*(g\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}) := (g \circ f) \cdot df_{i_1} \wedge \cdots \wedge df_{i_p}$$

for all $g \in \Omega^0(V)$ and $i_1, \dots, i_p \in \{1, \dots, m\}$.

Lemma 13.7 (Properties of pullbacks)**.** *Let $f\colon U \to V$ be a smooth map between open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, and let $\omega \in \Omega^p(V)$ be a p-form on V.*

(a) *For any q-form $\eta \in \Omega^q(V)$ we have $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.*

(b) *$df^*\omega = f^*(d\omega)$.*

(c) *If $g\colon W \to U$ is another smooth map from an open subset $W \subset \mathbb{R}^k$ then $g^*(f^*\omega) = (f \circ g)^*\omega$.*

(d) *If $m = n$ then $f^*(dx_1 \wedge \cdots \wedge dx_n) = \det f' \cdot dx_1 \wedge \cdots \wedge dx_n$.*

Proof. By linearity, it suffices to take $\omega = h\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ for a smooth function $h \in \Omega^0(V)$ and $i_1, \dots, i_p \in \{1, \dots, m\}$.

(a) This follows directly from the definition.

(b) First, in the case $p = 0$, i. e. for $\omega = h \in \Omega^0(V)$ a smooth function, we have by the multivariate chain rule [G1, Proposition 25.30]

$$\begin{aligned} df^*h = d(h \circ f) &= \sum_{i=1}^n \partial_i (h \circ f)\,dx_i = \sum_{i=1}^n \sum_{j=1}^m (\partial_j h \circ f) \cdot \partial_i f_j\,dx_i = \sum_{j=1}^m (\partial_j h \circ f) \cdot df_j \\ &= f^*\left(\sum_{j=1}^m \partial_j h \cdot dx_j \right) = f^*(dh). \end{aligned}$$

Now for arbitrary p, when computing $df^*\omega = d((h \circ f) \cdot df_{i_1} \wedge \cdots \wedge df_{i_p})$ we see by Lemma 13.4 that we can split the exterior derivative onto the factors, and leave out $d^2 f_{i_1}, \dots, d^2 f_{i_p}$. Hence, we obtain

$$df^*\omega = d(h \circ f) \wedge df_{i_1} \wedge \cdots \wedge df_{i_p} = f^*(dh) \wedge f^*dx_{i_1} \wedge \cdots \wedge f^*dx_{i_p} \overset{\text{(a)}}{=} f^*(d\omega).$$

(c) By (a) and (b), the pullbacks f^* and g^* commute with the wedge product and the exterior derivative. Hence, to check the statement for any $\omega = h\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ it suffices to check it for the functions $h, x_{i_1}, \dots, x_{i_p} \in \Omega^0(V)$, where it is obvious.

(d) By (a), we have

$$f^*(dx_1 \wedge \cdots \wedge dx_n) = df_1 \wedge \cdots \wedge df_n = \sum_{i_1, \dots, i_n = 1}^n \partial_{i_1} f_1 \cdot \cdots \cdot \partial_{i_n} f_n\,dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$

But $dx_{i_1} \wedge \cdots \wedge dx_{i_n}$ is $\operatorname{sign}(\pi) \cdot dx_1 \wedge \cdots \wedge dx_n$ if $(i_1, \dots, i_n)$ is a permutation π of $(1, \dots, n)$ and zero otherwise, so we obtain

$$f^*(dx_1 \wedge \cdots \wedge dx_n) = \sum_\pi \operatorname{sign}(\pi) \cdot \partial_{\pi(1)} f_1 \cdot \dots \cdot \partial_{\pi(n)} f_n \, dx_1 \wedge \cdots \wedge dx_n,$$

with the sum taken over all these permutations. By [G1, Remark 18.14], this is just the same as $\det f' \cdot dx_1 \wedge \cdots \wedge dx_n$. □

Example 13.8. Consider the map

$$f \colon \mathbb{R} \to \mathbb{R}^2, \, x \mapsto y := \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right)$$

with image $S^1 \backslash \{(0,1)\}$ that we saw already in Example 10.4 (b) as the inverse of a chart for S^1. Let us denote the coordinates in the target $\mathbb{R}^2$ by $y = (y_1, y_2)$.

(a) The pullback by f of the 1-form $\omega = y_1 \, dy_2$ on $\mathbb{R}^2$ is obtained as expected by the usual rules of substitution:

$$f^*\omega = \frac{2x}{x^2+1} \cdot d\left(\frac{x^2-1}{x^2+1} \right) = \frac{2x}{x^2+1} \cdot \frac{4x}{(x^2+1)^2} \, dx = \frac{8x^2}{(x^2+1)^3} \, dx.$$

(b) The pullback by f of any 2-form on $\mathbb{R}^2$ is clearly zero since there are non non-trivial 2-forms on $\mathbb{R}^1$.

We are now ready to carry over differential forms to corresponding vector bundles on manifolds.

Construction 13.9 (Differential forms on manifolds)**.** Let $(X, (\varphi_i)_{i \in I})$ be an n-dimensional manifold.

(a) For simplicity, let us first construct 1-forms on X. The idea is that on any chart $\varphi_i \colon U_i \to V_i$ with $i \in I$ we will just see a 1-form on $V_i \subset \mathbb{R}^n$ in the sense of Construction 13.1 (a), i.e. an expression of the form

$$\omega_i = (f_{i,1} \circ \varphi_i^{-1}) \, dx_1 + \cdots + (f_{i,n} \circ \varphi_i^{-1}) \, dx_n \quad \in \Omega^1(V_i)$$

for a smooth vector-valued map $f_i = (f_{i,1}, \dots, f_{i,n}) \colon U_i \to \mathbb{R}^n$, i.e. a section of the trivial bundle $U_i \times \mathbb{R}^n$ over U_i. Now if $\varphi_j \colon U_j \to V_j$ is another chart with $j \in I$, we have a similar representation of $\omega_j \in \Omega^1(V_j)$ by $f_j \colon U_j \to \mathbb{R}^n$, and as ω_i and ω_j are supposed to be different coordinate representations of the same 1-form on $U_i \cap U_j \subset X$, they should be related by the transition map $\varphi_i \circ \varphi_j^{-1}$ using the pullback of Definition 13.6. We thus require that $\omega_j = (\varphi_i \circ \varphi_j^{-1})^* \omega_i$ on the common intersection, which means in terms of f_i and f_j that

$$\begin{aligned} \sum_{k=1}^n (f_{j,k} \circ \varphi_j^{-1}) \, dx_k &= (\varphi_i \circ \varphi_j^{-1})^* \left(\sum_{l=1}^n (f_{i,l} \circ \varphi_i^{-1}) \, dx_l \right) \\ &= \sum_{l=1}^n (f_{i,l} \circ \varphi_j^{-1}) \, d(\varphi_i \circ \varphi_j^{-1})_l \\ &= \sum_{k,l=1}^n (f_{i,l} \circ \varphi_j^{-1}) \, \partial_k((\varphi_i \circ \varphi_j^{-1})_l) \, dx_k. \end{aligned}$$

Comparing the dx_k-coefficients and precomposing with φ_j, this means that $f_j = A_{i,j} \cdot f_i$ for the matrix-valued function

$$A_{i,j} \colon U_i \cap U_j \to \mathrm{GL}(n, \mathbb{R}), \, x \mapsto \left(\partial_k((\varphi_i \circ \varphi_j^{-1})_l)(\varphi_j(x)) \right)_{k,l} = ((\varphi_i \circ \varphi_j^{-1})')^{\mathsf{T}}(\varphi_j(x)). \qquad (*)$$

It is checked immediately that these functions satisfy the compatibility conditions $A_{i,i} = E$ and $A_{i,k} = A_{j,k} \cdot A_{i,j}$ of Construction 12.23 where defined for all $i, j, k \in I$, so as in Example 12.25 they define a vector bundle Ω_X^1 of rank n on X that has these transition maps. Note that the matrices $A_{i,j}(x)$ for $x \in U_i \cap U_j$ appearing here are just the transposed inverse matrices of the ones for the tangent bundle T_X in Example 12.25 (b). One can show that this means that

Ω^1_X is the *dual bundle* of T_X, obtained by making the dual vector spaces $T^\vee_{X,x}$ of the tangent spaces $T_{X,x}$ for $x \in X$ into a new vector bundle. We will not need this notion of dual bundles and therefore not go into details here, but just use this observation as a motivation why Ω^1_X is usually called the **cotangent bundle** of X.

(b) The construction of general p-forms for $p \in \mathbb{N}$ with $p \leq n$ is completely analogous. Again, a p-form on X will be given by a collection of p-forms $\omega_i \in \Omega^p(V_i)$ for all $i \in I$, satisfying the compatibility condition that $\omega_j = (\varphi_i \circ \varphi_j^{-1})^* \omega_i$ where defined for all $i, j \in I$. The only difference is that we now need $\binom{n}{p}$ instead of n coordinate functions to describe these forms, and hence that the transition maps will be more complicated when expressed in terms of matrix functions. Fortunately however, we do not need to know what they look like – the compatibility of pullbacks with wedge products of Lemma 13.7 (a) ensures that everything works out, and for actual computations we will just use the equation $\omega_j = (\varphi_i \circ \varphi_j^{-1})^* \omega_i$.

In this way we obtain **vector bundles Ω^p_X of p-forms** on X, of rank $\binom{n}{p}$. Of course, for $p = 0$ we just get the trivial bundle $\Omega^0_X = X \times \mathbb{R}$ of smooth functions on X, and for $p < 0$ or $p > n$ we set Ω^p_X to be the trivial vector bundle of rank 0.

(c) Having constructed the vector bundles Ω^p_X for all $p \in \mathbb{Z}$, a *differential form* of degree p (or *p-form*) on X is just defined to be a section of Ω^p_X over X. For simplicity, we will denote the vector space of all such p-forms on X by $\Omega^p(X)$ instead of $\Omega^p_X(X)$.

By (b), elements of $\Omega^p(X)$ can be given by choosing an atlas $(\varphi_i)_{i \in I}$ for X and specifying differential forms $\omega_i \in \Omega^p(V_i)$ for all charts $\varphi_i \colon U_i \to V_i \subset \mathbb{R}^n$ such that the compatibility condition $\omega_j = (\varphi_i \circ \varphi_j^{-1})^* \omega_i$ holds for all $i, j \in I$. In particular, if X is an open subset of $\mathbb{R}^n$ (for which only one chart is needed) our new definition of $\Omega^p(X)$ coincides with the old one from Construction 13.1 (a).

(d) Finally, note that wedge products and exterior derivatives commute with pullbacks by Lemma 13.7, and thus with the compatibility condition $\omega_j = (\varphi_i \circ \varphi_j^{-1})^* \omega_i$. Hence, we obtain well-defined bilinear *wedge products* $\Omega^p(X) \times \Omega^q(X) \to \Omega^{p+q}(X)$ and *exterior derivatives* $d \colon \Omega^p(X) \to \Omega^{p+1}(X)$ for all $p, q \in \mathbb{Z}$ on any manifold X, given on coordinate charts by Construction 13.1 (b) and Definition 13.3.

Remark 13.10.

(a) Clearly, the properties of the exterior derivative in Lemma 13.4 carry over to differential forms on manifolds since they can be checked locally on coordinate charts.

(b) Similarly to Remark 13.2 (a), there is actually a general construction of *alternating tensor products* for vector bundles, and in this sense Ω^p_X is just the p-th alternating tensor product of the cotangent bundle Ω^1_X. But again, we will not use this general construction or terminology in these notes.

(c) Note that the exterior derivative is a map on sections $d \colon \Omega^p(X) \to \Omega^{p+1}(X)$ for any manifold X, but clearly not a morphism of vector bundles $\Omega^p_X \to \Omega^{p+1}_X$: By definition, such a morphism would have to be a linear map in each fiber, but differentiation *is not even defined in fibers* as the value of a derivative $f'(x)$ at a point $x \in X$ does not only depend on $f(x)$, but on the values of f in a neighborhood of x.

However, as a map on sections the exterior derivative is even a contravariant functor:

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Construction 13.11 (Differential forms are functorial)**.** Let $f \colon (X, (\varphi_i)_{i \in I}) \to (Y, (\psi_j)_{j \in J})$ be a morphism of manifolds. Then for any differential form $\omega \in \Omega^p(Y)$ we can construct a *pullback* $f^*\omega \in \Omega^p(X)$ using Definition 13.6 on coordinate charts: For any charts $\varphi_i \colon U_i \to V_i$ and $\psi_j \colon U'_j \to V'_j$ with $f(U_i) \subset U'_j$ the form ω is given in coordinates on V'_j by some $\omega_j \in \Omega^p(V'_j)$, and we assign to it the pullback $(\psi_j \circ f \circ \varphi_i^{-1})^* \omega_j \in \Omega^p(V_j)$ in the sense of Definition 13.6. It is checked immediately using Lemma 13.7 (c) that this is compatible with coordinate changes in the source and target, hence gives rise to a well-defined differential form $f^*\omega \in \Omega^p(X)$.

Moreover, Lemma 13.7 (c) confirms that this assignment $X \mapsto \Omega^p(X)$ is functorial for morphisms of manifolds for all p, making it into a contravariant functor $\Omega^p : \mathsf{Man} \to \mathsf{Vect}$. By Lemma 13.7 (a) and (b), it commutes with wedge products and the exterior derivative.

Note that the pullback $f^* : \Omega^p(Y) \to \Omega^p(X)$ is particularly easy to construct for embeddings $X \to Y$ of submanifolds: In this case, it is just obtained by restricting sections of the vector bundle Ω^p_Y to X as in Construction 12.4 (a). Hence, in this case we will often write a pullback $f^*\omega$ as $\omega|_X$, or even just ω if it is clear that we are considering the form only on X.

Example 13.12. Consider the manifold $X = S^1 \subset \mathbb{R}^2$ with the charts $\varphi_1 : U_1 = X\backslash\{(0,1)\} \to V_1 = \mathbb{R}$ and $\varphi_2 : U_2 = X\backslash\{(0,-1)\} \to V_2 = \mathbb{R}$ as in Example 10.4 (b). Moreover, let S^1 be embedded as usual in the plane $\mathbb{R}^2$, whose coordinates we denote by $y = (y_1, y_2)$.

(a) The restriction $\omega := y_1\, dy_2|_{S^1}$ of the form $y_1\, dy_2 \in \Omega^1(\mathbb{R}^2)$ is a well-defined 1-form on S^1. In the local coordinate x_1 on $V_1 = \mathbb{R}$ it is given by

$$\omega_1 := (\varphi_1^{-1})^*(y_1\, dy_2) \overset{13.8}{=} \frac{8x_1^2}{(x_1^2+1)^3}\, dx_1.$$

Similarly, one can compute its representation in the local coordinate x_2 on $V_2 = \mathbb{R}$ to be $\omega_2 = -\frac{8x_2^2}{(x_2^2+1)^3}\, dx_2$.

(b) Conversely, we could define the same form $\omega \in \Omega^1(S^1)$ without referring to the ambient $\mathbb{R}^2$ by only giving the two explicit formulas above for ω_1 and ω_2 on V_1 and V_2, respectively, noting that these two local representations are compatible since the transition map is given by $\varphi_2 \circ \varphi_1^{-1} : V_1\backslash\{0\} \to V_2\backslash\{0\},\ x_1 \mapsto x_2 = \frac{1}{x_1}$ by Example 10.4 (b), and

$$(\varphi_2 \circ \varphi_1^{-1})^*\omega_2 = -\frac{8(\frac{1}{x_1})^2}{((\frac{1}{x_1})^2+1)^3}\, d(\tfrac{1}{x_1}) = -\frac{8(\frac{1}{x_1})^2}{((\frac{1}{x_1})^2+1)^3} \cdot \frac{-1}{{x_1}^2}\, dx_1 = \omega_1.$$

Exercise 13.13. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a submersion of manifolds, and let $X := f^{-1}(\{0\})$ be the submanifold given as the vanishing set of f. For all $i = 1, \ldots, n$ we set

$$\omega_i := (-1)^{i-1} \left(\frac{\partial f}{\partial x_i}\right)^{-1} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where $\widehat{\cdot}$ means that the corresponding term has been omitted.

(a) For all $i, j = 1, \ldots, n$, show that $\omega_i = \omega_j$ at the points of X where both forms are defined.

(b) Use (a) to check directly that there is a nowhere-zero $(n-1)$-form on X.

As announced, we can now use Construction 13.9 to set up an analytic version of cohomology.

Construction 13.14 (De Rham cohomology)**.** Let X be a manifold.

(a) By Lemma 13.4 (a) and Remark 13.10 (a) there is a cochain complex

$$\cdots \longrightarrow \Omega^{p-1}(X) \xrightarrow{d} \Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X) \longrightarrow \cdots$$

of differential forms on X. It is called the **de Rham cochain complex** of X and denoted by $\Omega(X)$. By Construction 13.11, the assignment $X \mapsto \Omega(X)$ is a contravariant functor from Man to $\mathsf{Ch}^\vee$, hence we call it the **de Rham cochain functor**.

(b) As for general cochain complexes, differential forms in the kernel of d will be called **closed** or **cocycles**, and forms in the image of d will be called **coboundaries**. In de Rham theory, it is also common to call such coboundaries **exact** differential forms.

(c) The cohomology groups of the de Rham cochain complex

$$H^p_{\mathrm{dR}}(X) := H^p(\Omega(X))$$

are called the **de Rham cohomology groups** of X; hence in the wording of (b) the elements of $H^p_{\mathrm{dR}}(X)$ are closed p-forms modulo exact p-forms on X. As usual, the dimensions $b^p_{\mathrm{dR}}(X) = \dim H^p_{\mathrm{dR}}(X) \in \mathbb{N}\cup\{\infty\}$ are called the **de Rham Betti numbers** of X.

As the composition of the de Rham cochain functor with the cohomology functor, the assignment $X \mapsto H^p_{\mathrm{dR}}(X)$ is clearly a contravariant functor $\mathsf{Man} \to \mathsf{Vect}$. We will see in Proposition 15.10 that $H^p_{\mathrm{dR}}(X)$ is naturally isomorphic to the singular cohomology group $H^p(X)$ for the ground field $\mathbb{R}$ (so that in particular we obtain the same Betti numbers $b^p_{\mathrm{dR}}(X) = b_p(X)$ as in the singular setting) – but as long as we have not shown this we will use the index "dR" to distinguish between the two groups and Betti numbers.

Example 13.15.

(a) If a manifold X has finitely many connected components $X_1,\dots,X_n$, we clearly have $\Omega^p(X) = \Omega^p(X_1)\times\cdots\times\Omega^p(X_n)$ for all $p\in\mathbb{Z}$ as giving a p-form on X is the same as giving a p-form independently on each component. Also, the exterior differential $d\colon \Omega^p(X)\to\Omega^{p+1}(X)$ is just the product of the exterior differentials on the components, and thus we see that $H^p_{\mathrm{dR}}(X)\cong H^p_{\mathrm{dR}}(X_1)\times\cdots\times H^p_{\mathrm{dR}}(X_n)$.

(b) A smooth function $f\in\Omega^0(X)$ on a manifold X satisfies $df = 0\in\Omega^1(X)$ if and only if it is locally constant: As this is a local statement we may assume that $X\subset\mathbb{R}^n$ is an open subset, in which case

$$df = \partial_1 f\,dx_1+\cdots+\partial_n f\,dx_n \overset{!}{=} 0$$

is equivalent to $f' = 0$, which exactly means that f is locally constant. In particular, if X is connected then the closed 0-forms are just the constant functions. As 0-forms can never be exact, we conclude that $H^0_{\mathrm{dR}}(X)\cong\mathbb{R}$ in this case, i. e. that $b^0_{\mathrm{dR}}(X)=1$. If X is not necessarily connected, then by (a) the Betti number $b^0_{\mathrm{dR}}(X)$ is the number of connected components of X.

Of course, this just corresponds to the analogous statement for singular (co-)homology in Remark 6.10. In case you are surprised about *connected* instead of *path-connected* components showing up here: Connected components of X are the natural setting for functions from X to $\mathbb{R}$ (which we have here – such a function is locally constant if and only if it is constant on each *connected* component of X), whereas path-connected components are the natural setting for functions from $\mathbb{R}$ to X (which are needed in singular homology). For manifolds however, this does not make a difference by Exercise 10.12 (a).

(c) Let $X=(a,b)\subset\mathbb{R}$ be an open interval in $\mathbb{R}$. Clearly, we have $b^p_{\mathrm{dR}}(X)=0$ for $p<0$ or $p>1$, and $b^0_{\mathrm{dR}}(X)=1$ by (b). To compute $b^1_{\mathrm{dR}}(X)$, note that every $\omega\in\Omega^1(X)$ is of the form $f\,dx$ for a smooth function f, and by integrating f we can clearly find a smooth function $F\in\Omega^0(X)$ such that $dF = F'\,dx = f\,dx$. Hence, every 1-form on X is exact, which means that $b^1_{\mathrm{dR}}(X)=0$ (as expected from topology).

(d) Recalling the initial motivating example from the introduction to these notes, the "derivative of the angle in polar coordinates"

$$\omega = d\arctan\frac{x_2}{x_1} = -\frac{x_2}{x_1^2+x_2^2}\,dx_1 + \frac{x_1}{x_1^2+x_2^2}\,dx_2 \qquad \in\Omega^1(\mathbb{R}^2\backslash\{0\})$$

is closed since $d^2=0$, but not exact: Assume for a contradiction that there is a smooth function $f\in\Omega^0(\mathbb{R}^2\backslash\{0\})$ with $df=\omega$. Then on $(\mathbb{R}\backslash\{0\})\times\mathbb{R}$, where the angle function $g\colon x\mapsto\arctan\frac{x_2}{x_1}$ is well-defined, we have $dg=\omega$ as well, and thus $d(f-g)=0$. As in (b) this means that $f-g$ is locally constant, i. e. up to an additive constant f must be the angle function on $\mathbb{R}_{>0}\times\mathbb{R}$ and $\mathbb{R}_{<0}\times\mathbb{R}$ – but there is no continuous way to make this into a function on all of $\mathbb{R}^2\backslash\{0\}$.

Hence, we see that ω is a non-zero element in $H^1_{\mathrm{dR}}(\mathbb{R}^2\backslash\{0\})$, showing that $b^1_{\mathrm{dR}}(\mathbb{R}^2\backslash\{0\})\ge 1$. Of course, this is just the differential form corresponding to the non-zero simplicial cocycle in the annulus in Example 5.11, and once we know that de Rham cohomology coincides with singular cohomology it will be clear that $b^1_{\mathrm{dR}}(\mathbb{R}^2\backslash\{0\}) = 1$, detecting the one hole in the punctured plane.

Remark 13.16 (Ring structure on de Rham cohomology)**.** Note that the wedge product of Construction 13.1 (b) passes to a well-defined product

$$H^p_{\mathrm{dR}}(X) \times H^q_{\mathrm{dR}}(X) \to H^{p+q}_{\mathrm{dR}}(X),\ (\omega,\eta) \mapsto \omega \wedge \eta$$

on any manifold X: If $\omega \in \Omega^p(X)$ and $\eta \in \Omega^q(X)$ are closed then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p\, \omega \wedge d\eta = 0$$

is closed as well by Lemma 13.4 (b), and if in addition e. g. $\omega = d\varphi$ is exact then by the same lemma

$$\omega \wedge \eta = d\varphi \wedge \eta = d\varphi \wedge \eta + (-1)^{p-1}\, \varphi \wedge d\eta = d(\varphi \wedge \eta)$$

is also exact. This makes the product of all $H^p_{\mathrm{dR}}(X)$ for $p \in \mathbb{Z}$ into a (graded, associative, but non-commutative) ring. Note that this product resp. ring structure on de Rham cohomology appears very naturally, in contrast to simplicial cohomology (where a corresponding structure could also be defined, but with much more effort). We will see the geometric meaning of this ring structure in Proposition 16.11.