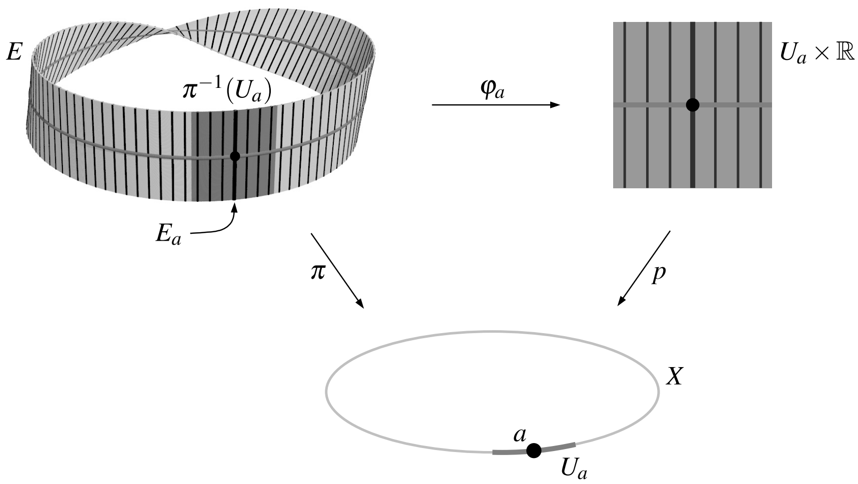
12. Vector Bundles

In this chapter we want to study so-called vector bundles, a certain kind of manifolds with an extra structure that often occurs in practice. The main motivation probably comes from the study of tangent vectors and tangent vector fields that we considered already in the Hairy Ball Theorem in Proposition 9.5: First of all, given an n-dimensional submanifold X of $\mathbb{R}^m$, we want to define the *tangent space* $T_{X,a}$ to X at any point $a \in X$ as the n-dimensional linear subspace of $\mathbb{R}^m$ which, when shifted to a, approximates X best around a. We then also want to consider *tangent vector fields*, i. e. maps that associate to any $a \in X$ a tangent vector at this point in a smooth way. Note that to be able to define what smoothness means in this situation, it is not sufficient to construct a tangent space separately at each point – instead, in Construction 12.13 we will make the union $T_X := \bigcup_{a\in X} \left(\{a\} \times T_{X,a}\right)$ of all tangent spaces into a global $2n$-dimensional manifold called the *tangent bundle*, so that a tangent vector field on X can be defined as a *smooth* map from X to T_X that assigns to every $a \in X$ a point in $T_{X,a}$.

It might be less geometrically obvious, but we will see later in Example 12.25 that linear approximations of a manifold and hence the tangent bundle can in fact also be defined abstractly for arbitrary manifolds, i. e. not only for submanifolds of $\mathbb{R}^m$. In fact, our setting of assigning smoothly varying tangent spaces to points of a manifold is just a special case of a very general concept called a *vector bundle*, which assigns to every point of a manifold a smoothly varying vector space.

Example 12.1 (The Möbius strip as a vector bundle)**.** To consider a completely different and easier example illustrating this concept, let E be the Möbius strip shown in the top left of the picture below. In contrast to Example 1.17 (c), let us remove its boundary points to make it into a 2-dimensional manifold, and let us consider the dark vertical lines to be the real line $\mathbb{R}$ instead of a bounded interval (which does not make a topological difference since these spaces are homeomorphic). The Möbius strip E then comes with a natural projection map $\pi\colon E \to X$ to its center line $X = S^1$, and the inverse image $E_a := \pi^{-1}(\{a\})$ of every point $a \in X$ is a 1-dimensional vector space (namely one of the vertical lines).



Note that this assignment of a 1-dimensional vector space E_a to every $a \in X$ is *locally trivial* in the sense that for every such point a there is an open neighborhood U_a (shown in dark gray in the picture) such that $\pi^{-1}(U_a)$ just looks like $U_a \times \mathbb{R}$, i. e. like a non-varying vector space $\mathbb{R}$ sitting at every point of U_a. It is only the *global structure* of E that is different, since running around the circle

X will exchange the positive and negative part of $E_a \cong \mathbb{R}$. We will see a precise definition of this bundle in Construction 12.15.

Actually, the same phenomenon occurs e. g. for the tangent bundle of the 2-sphere S^2 in the Hairy Ball Theorem of Proposition 9.5: Locally, we could flatten out the sphere so that its tangent spaces are the same at every point, but globally the tangent planes will wrap around the sphere and thus form a non-trivial varying family of 2-dimensional vector spaces. Let us now describe this situation formally, using the notations from the picture above.

Construction 12.2 (Vector bundles)**.** Let X be an n-dimensional manifold. A (real) **vector bundle** E of **rank** $\operatorname{rk} E = r \in \mathbb{N}$ on X is a manifold E of dimension $n+r$ together with a morphism $\pi\colon E \to X$ and a real vector space structure on the **fiber** $E_x := \pi^{-1}(\{x\}) \subset E$ for every $x \in X$, satisfying the following condition:

Any point $a \in X$ admits an open neighborhood $U_a \subset X$ and an isomorphism of manifolds $\varphi_a\colon \pi^{-1}(U_a) \to U_a \times \mathbb{R}^r$ such that the following two properties hold.

(a) On $\pi^{-1}(U_a)$ we have $\pi = p \circ \varphi_a$, where $p\colon U_a \times \mathbb{R}^r \to U_a$ is the projection onto the first factor (i. e. the diagram in the picture above commutes). In other words, φ_a maps the fiber $E_x \subset E$ to the fiber $\{x\} \times \mathbb{R}^r \subset U_a \times \mathbb{R}^r$, which is often expressed by saying that φ_a is a morphism **over** U_a and written as $\varphi_a|_{E_x}\colon E_x \to \mathbb{R}^r$, identifying $\{x\} \times \mathbb{R}^r$ with $\mathbb{R}^r$. Of course, as φ_a is an isomorphism, $\varphi_a|_{E_x}$ is bijective.

(b) This bijection $\varphi_a|_{E_x}\colon E_x \to \mathbb{R}^r$ is linear, i. e. an isomorphism of vector spaces, for all $x \in U_a$.

Instead of saying that E is a vector bundle on X with the data given above, one also often says that the morphism $\pi\colon E \to X$ is a vector bundle, and that X is the **base** of the bundle.

Example 12.3 (Trivial bundles)**.** For any manifold X and $r \in \mathbb{N}$, the manifold $E = X \times \mathbb{R}^r$ together with the morphism $\pi\colon E \to X$, $(x,v) \mapsto x$ and the identity maps $\varphi_a\colon X \times \mathbb{R}^r \to X \times \mathbb{R}^r$ for any $a \in X$ is a vector bundle of rank r over X. It is called the **trivial vector bundle** of rank r over X since it corresponds to a non-varying vector space $\mathbb{R}^r$ sitting at every point of X.

With this notation, one could say intuitively that the isomorphisms $\varphi_a\colon \pi^{-1}(U_a) \to U_a \times \mathbb{R}^r$ certify that E is locally isomorphic to the trivial vector bundle of rank r around each point $a \in X$. In fact, the maps φ_a are called **local trivializations** of E for this reason. Let us now make this into a precise statement by defining restrictions and morphisms of vector bundles in the expected way. 19

Construction 12.4 (Restrictions and morphisms of vector bundles)**.** Let $\pi\colon E \to X$ be a vector bundle with $\dim X = n$ and $\operatorname{rk} E = r$.

(a) Let $Y \subset X$ be a submanifold. Note that the map π is a submersion: This can be checked locally, so by Remark 10.18 (a) we may assume for any $a \in X$ that we have a local trivialization $\varphi_a\colon \pi^{-1}(U_a) \to U_a \times \mathbb{R}^r$ for an open subset $U_a \subset \mathbb{R}^n$, which we can then take as a chart for E. But in this chart, the map $\pi \circ \varphi_a^{-1}$ is just given by $U_a \times \mathbb{R}^r \to U_a$, $(x,v) \mapsto x$, which clearly has rank $n = \dim X$.

Hence, by Proposition 11.9 (a) the inverse image $E|_Y := \pi^{-1}(Y)$ is a submanifold of E. Note that the projection π restricts to a morphism $\pi|_{E|_Y}\colon E|_Y \to Y$ whose fiber over $a \in Y$ is just the vector space E_a. Moreover, for $a \in Y$ we can restrict the local trivialization maps $\varphi_a\colon \pi^{-1}(U_a) \to U_a \times \mathbb{R}^r$ of E to local trivializations $\pi^{-1}(U_a \cap Y) \to (U_a \cap Y) \times \mathbb{R}^r$ of $E|_Y$, making $E|_Y$ into a vector bundle of rank r on Y. We call it the **restriction** of E to Y.

(b) A **morphism** of E to another vector bundle $\psi\colon F \to X$ over the same base X is a morphism of manifolds $f\colon E \to F$ over X (i. e. with $\psi \circ f = \varphi$, such that f maps the fiber E_x to the fiber F_x for all $x \in X$) such that all maps $f|_{E_x}\colon E_x \to F_x$ are linear for the given vector space structures on the fibers. We denote the set of all such morphisms by $\operatorname{Hom}_X(E,F)$, or just $\operatorname{Hom}(E,F)$ if X is clear from the context.

In particular, this makes vector bundles over X into a category, and defines the notion of isomorphisms of vector bundles over X. Combining this with (a), Construction 12.2 just says that a vector bundle is a morphism $\pi\colon E \to X$, with a given vector space structure in each fiber, and the requirement that it is locally isomorphic to a trivial bundle.

Combining this with Remark 10.18 (a), this means in practice that for any purposes local on X we can always assume that a vector bundle is just the projection $\pi\colon U\times\mathbb{R}^r \to U$ of a trivial vector bundle of rank r over an open subset $U \subset \mathbb{R}^n$.

Remark 12.5.

(a) The definition of a vector bundle is very similar to that of a covering space in [G3, Chapter 8], with the only difference being that the fibers of a covering space are discrete topological spaces, whereas the fibers of a vector bundle are real vector spaces. Note that this vector space structure is actually important here, i. e. we do not only consider a vector bundle E over a base X as a topological space or manifold. In fact, from a purely topological point of view vector bundles are not very interesting: By linearly moving every point to the origin in each fiber, the base X is always a deformation retract of E, so that the singular homology of E is naturally isomorphic to that of X.

(b) Note that the vector space structure on the fibers of a vector bundle $\pi\colon E \to X$ of rank r is part of the fixed data that needs to be given to define a vector bundle, whereas the local trivializations $\varphi_a\colon \pi^{-1}(U_a) \to U_a \times \mathbb{R}^r$ are not – they just need to exist to guarantee that the vector space structure of the fibers varies smoothly, but similarly to charts of manifolds and bases of vector spaces they are not uniquely determined. The following construction shows what this means exactly.

Construction 12.6 ($\mathrm{Hom}(E,F)$ as a vector space)**.** Let $\pi\colon E\to X$ and $\psi\colon F \to X$ be vector bundles of rank r and k, respectively, on an n-dimensional manifold X.

(a) Clearly, for any two morphisms $f,g \in \mathrm{Hom}(E,F)$ and $\lambda\in\mathbb{R}$ there are linear pointwise addition and scalar multiplication maps in the fibers

$$E_x \to F_x,\ v \mapsto f|_{E_x}(v) + g|_{E_x}(v) \quad\text{and}\quad E_x \to F_x,\ v\mapsto \lambda f|_{E_x}(v)$$

that can be combined for all $x\in X$ into set-theoretic maps $f+g\colon E\to F$ and $\lambda f\colon E \to F$ over X. It now follows from the local triviality condition that these two maps are in fact smooth, hence also morphisms between E and F, and thus make $\mathrm{Hom}(E,F)$ into a real vector space: As smoothness is a local condition, we may assume by Construction 12.4 (b) that the bundles $\pi\colon U\times\mathbb{R}^r\to U$ and $\psi\colon U\times\mathbb{R}^k \to U$ are trivial with $U\subset\mathbb{R}^n$ open. The two morphisms of vector bundles are then given by $f\colon U\times\mathbb{R}^r \to U\times\mathbb{R}^k,\ (x,v)\mapsto (x,\tilde f(x,v))$ and $g\colon U\times\mathbb{R}^r\to U\times\mathbb{R}^k,\ (x,v)\mapsto(x,\tilde g(x,v))$ for smooth functions $\tilde f,\tilde g\colon U\times\mathbb{R}^r\to\mathbb{R}^k$. But then

$$\begin{aligned} f+g\colon U\times\mathbb{R}^r &\to U\times\mathbb{R}^k,\ (x,v)\mapsto (x,\tilde f(x,v)+\tilde g(x,v)) \\ \text{and}\qquad \lambda f\colon U\times\mathbb{R}^r &\to U\times\mathbb{R}^k,\ (x,v)\mapsto(x,\lambda\tilde f(x,v))\end{aligned}$$

are also smooth since smooth $\mathbb{R}^k$-valued functions on open subsets of $\mathbb{R}^n\times\mathbb{R}^r$ form a vector space. This is exactly what it means that "the vector space structure in a bundle varies smoothly with the point".

(b) Expanding the idea of (a), the map $\tilde f\colon U\times\mathbb{R}^r\to\mathbb{R}^k$ is smooth and linear in the second component, and thus by linear algebra must be of the form

$$\tilde f\colon U\times\mathbb{R}^r\to\mathbb{R}^k,\ (x,v)\mapsto A(x)\cdot v$$

for a unique smooth matrix-valued map $A\colon U\to\mathbb{R}^{k\times r},\ x\mapsto A(x)$. In other words, in given local trivializations a morphism $E\to F$ of vector bundles is just the same as a morphism of manifolds with target space $\mathbb{R}^{k\times r}$. In particular, a morphism between the trivial bundles $X\times\mathbb{R}^r$ and $X\times\mathbb{R}^k$ is globally the same as a morphism $X\to\mathbb{R}^{k\times r}$.

(c) A consequence of this is that – as for vector spaces, but unlike the case of manifolds – any bijective morphism of vector bundles is an isomorphism: If f is bijective then the matrix $A(x)$ of (b) must be invertible at any point, hence we must have $k=r$ and obtain an inverse morphism $A^{-1}: U \to \mathbb{R}^{r\times r}$, $x \mapsto A(x)^{-1}$ (which is smooth by the formula for the inverse matrix in [G1, Proposition 18.21]), showing that the inverse map $f^{-1}: F \to E$ is in fact a morphism of vector bundles.

Note that this shows again that vector bundles can be thought of as vector spaces varying smoothly with the point on a manifold: For (finite-dimensional) vector spaces, a linear map is given by a matrix after a choice of bases. For vector bundles, a morphism is given by a matrices varying smoothly with the point on a manifold after a choice of local trivializations.

Consequently, we would also expect that we can apply standard constructions for linear maps to morphisms of vector bundles fiber-by-fiber, e. g. taking their images and kernels to obtain new vector bundles that are subsets of old ones. There is a slight complication here however since vector bundles must have constant rank, i. e. we can only hope for a matrix-valued function to determine image and kernel vector bundles if all matrices have the same rank. As vector bundles are not the main focus of this class, let us not go into details here, but just consider injective morphisms leading to image subbundles of the target bundle in the following sense.

Definition 12.7 (Subbundles)**.** A **(vector) subbundle** of a vector bundle $\pi: E \to X$ is a vector bundle of the form $\pi|_F: F \to X$ for a submanifold $F \subset E$, with the vector space structure of F_a taken from E_a (so in particular, F_a has to be a vector subspace of F_a for all $a \in X$).

Lemma 12.8 (Image subbundles)**.** *Let X be an n-dimensional manifold, and consider a morphism $A: X \to \mathbb{R}^{k\times r}$, $x \mapsto A(x)$ for $k, r \in \mathbb{N}_{>0}$ with $\operatorname{rk} A(x) = r$ for all $x \in X$. Then the* ***image subbundle***

$$\operatorname{im} A := \bigcup_{x\in X} \big(\{x\} \times \operatorname{im} A(x)\big) = \{(x, A(x)\cdot v) : x \in X, v \in \mathbb{R}^r\} \quad \subset X \times \mathbb{R}^k$$

is a subbundle of the trivial bundle $X \times \mathbb{R}^k$ which is isomorphic to the trivial bundle $X \times \mathbb{R}^r$.

Proof. As $\operatorname{rk} A(x) = r$ the matrix $A(x)^{\mathsf{T}} A(x) \in \mathbb{R}^{r\times r}$ is invertible for all $x \in X$, and thus we can construct the morphism

$$B: X \to \mathbb{R}^{r\times k},\ x \mapsto \big(A(x)^{\mathsf{T}} A(x)\big)^{-1} A(x)^{\mathsf{T}}$$

(note that the explicit formula for the inverse matrix [G1, Proposition 18.21] shows that B is in fact smooth); it is a left inverse for A in the sense that $B(x)A(x) = E$ for all $x \in X$. Now consider the corresponding morphisms of trivial vector bundles

$$f: X \times \mathbb{R}^r \to X \times \mathbb{R}^k,\ (x,v) \mapsto (x, A(x)\cdot v) \quad \text{and} \quad g: X \times \mathbb{R}^k \to X \times \mathbb{R}^r,\ (x,v) \mapsto (x, B(x)\cdot v)$$

as in Construction 12.6 (b). Clearly, the map f has image $\operatorname{im} A$, and as g is a left inverse we see that $f: X \times \mathbb{R}^r \to \operatorname{im} A$ must be a homeomorphism (with inverse $g|_{\operatorname{im} A}$). Moreover, $g \circ f = \mathrm{id}$ requires the rank of f in the sense of Construction 10.5 (b) to be $n + r$, so f is an immersion, and thus $f: X \times \mathbb{R}^r \to \operatorname{im} A$ is an isomorphism of manifolds by Proposition 11.9 (b). Finally, both f and g are linear in v, so g is a global trivialization of $\operatorname{im} A$ and thus makes it into a subbundle of $X \times \mathbb{R}^k$ isomorphic to $X \times \mathbb{R}^r$. □

Remark 12.9 (Local image subbundles)**.**

(a) The important point of Lemma 12.8 is not that $\operatorname{im} A$ is trivial, but that the smoothness of the morphism A ensures that the vector subspaces of the fibers determined by $\operatorname{im} A$ vary smoothly, making $\operatorname{im} A$ into a subbundle. Actually, being a subbundle is a local condition on the base, so if $\pi: E \to X$ is a vector bundle then any subset $F \subset E$ will also be a subbundle if there is an open cover $(U_i)_{i\in I}$ of X on which E is trivial and such that $\pi^{-1}(U_i) \cap F$ is an image subbundle of $\pi^{-1}(U_i)$ over U_i as in Lemma 12.8 in suitable local trivializations for all $i \in I$ – and due to the gluing process by the cover and the local trivializations the subbundle F of E obtained in this way will then in general be non-trivial.

(b) In particular, this situation occurs if we have an injective morphism $f: F \to E$ of vector bundles over a manifold X. Then the maps $f|_{E_x}: E_x \to F_x$ in the fibers over any $x \in X$ are injective as well, so in local trivializations of E and F over a cover $(U_i)_{i\in I}$ the morphism f

is given by functions $U_i \to \mathbb{R}^{k\times r}$ mapping into the space of matrices of rank r. Hence, the image of f is a subbundle of E (which can be thought of as a relative version of lemma 12.8.

Clearly, this applies if $F \subset E$ is already known to be a subbundle in the first place and f is the inclusion; hence we also see conversely that any subbundle of a vector bundle is always locally an image subbundle in the sense of Lemma 12.8.

Exercise 12.10 (Kernel subbundle)**.** Let $f\colon E \to F$ be a morphism between two vector bundles $\pi\colon E \to X$ and $\psi\colon F \to X$ over a manifold X. We denote by $\ker f := \bigcup_{x\in X} \ker(f|_{E_x}) \subset E$ the union of the kernels of the linear maps in the fibers given by f.

(a) Show that $\ker f$ is a subbundle of E if and only if the dimension of the vector spaces $\ker(f|_{E_x})$ for $x \in X$ is constant.

(b) Give an example of a morphism f for which $\ker f$ is not a subbundle of E.

Exercise 12.11 (Pullbacks of bundles)**.** Let $f\colon X \to Y$ be a morphism of manifolds, and let $\pi\colon E \to Y$ a vector bundle on Y of rank r. We set

$$f^*(E) := \{(x,v) \in X \times E : f(x) = \pi(v)\} \quad \subset X \times E.$$

(a) Show that $f^*(E)$ is a submanifold of $X \times E$, and that the natural projection $\pi\colon f^*(E) \to X$ onto the first factor is a smooth map with $\pi^{-1}(\{x\}) = E_{f(x)}$ for all $x \in X$.

(b) Show that $\pi\colon f^*(E) \to X$ is a rank-r vector bundle. It is called the *pullback bundle* of E by f.

Exercise 12.12. Show that every vector bundle on a compact manifold is a subbundle of a trivial bundle.

We can now use the idea of Remark 12.9 to give many interesting vector bundle constructions. Let us start with the tangent bundle already mentioned at the beginning of this chapter.

Construction 12.13 (Tangent bundles and derivatives of morphisms)**.** Let X be a k-dimensional manifold. For simplicity, let us assume for the moment that it comes as a submanifold $X \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. Recall that by the Whitney embedding of Proposition 11.21 and Remark 11.22 (b) this is not a restriction except that it makes a possibly unnatural choice. We will study how to avoid this choice in Example 12.25.

(a) For any $x \in X$ we have a chart $\varphi\colon U \to V \subset \mathbb{R}^k$ around x, whose inverse $\varphi^{-1}\colon V \to U \subset \mathbb{R}^n$ we can consider as a local parametrization for X. The linearization of this map is its Jacobian matrix $A_\varphi(x) := (\varphi^{-1})'(\varphi(x)) \in \mathbb{R}^{n\times k}$, hence we define the **tangent space** of X at x to be the k-dimensional vector space

$$T_{X,x} := \operatorname{im} A_\varphi(x) \quad \subset \mathbb{R}^n.$$

Note that this is independent of the chart φ: A different choice would just precompose the Jacobian matrix $A_\varphi(x)$ with the (invertible) Jacobian matrix of the transition map, which does not change its image. We now collect these tangent spaces in a global object

$$T_X := \bigcup_{x\in X} (\{x\} \times T_{X,x}) \quad \subset X \times \mathbb{R}^n.$$

Locally on a chart $\varphi\colon U \to V$, this is just the image subbundle $\operatorname{im} A_\varphi$ of Lemma 12.8, so Remark 12.9 (a) shows that T_X is a subbundle of rank k of $X \times \mathbb{R}^n$ with fibers $T_{X,x}$. We call
20 it the **tangent bundle** of X.

(b) Tangent spaces can be used to construct well-defined derivatives of a morphism $f\colon X \to Y$ between manifolds for any submanifold $Y \subset \mathbb{R}^m$: For a point $x \in X$, choose charts $\varphi\colon U \to V$ and $\psi\colon U' \to V'$ with $f(U) \subset U'$ around x and $y := f(x)$, respectively, and consider the linear map

$$f'(x)\colon T_{X,x} \to T_{Y,y},\ (\varphi^{-1})'(\varphi(x)) \cdot v \mapsto (f \circ \varphi^{-1})'(\varphi(x)) \cdot v.$$

Note that:

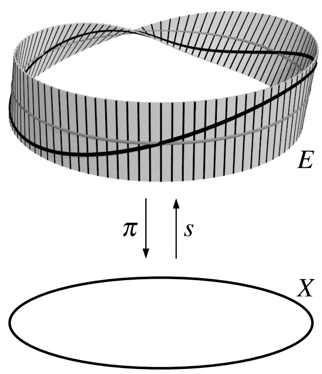
- Every vector in $T_{X,x}$ is of the form $(\varphi^{-1})'(\varphi(x)) \cdot v$ for a unique $v \in \mathbb{R}^k$, so $f'(x)$ is defined at every point of $T_{X,x}$. Moreover, the definition is independent of the choice of chart φ since a different choice would multiply both $(\varphi^{-1})'(\varphi(x))$ and $(f \circ \varphi^{-1})'(\varphi(x))$ with the same Jacobian matrix of the transition map.
- The definition is visibly independent of the chart ψ. However, note that the image of $T_{X,x}$ actually lies in $T_{Y,y}$ since

$$(f \circ \varphi^{-1})'(\varphi(x)) \cdot v = (\psi^{-1})'(\psi(y)) \cdot (\psi \circ f \circ \varphi^{-1})'(\varphi(x)) \cdot v \in T_{Y,y}.$$

We can think of $f'(x)$ as the linearization of f around x, mapping the linearization $T_{X,x}$ of X to the linearization $T_{Y,y}$ of Y. It is called the **derivative** of f at x. Varying the point, we can also consider it as a morphism of manifolds $f' : T_X \to T_Y$ (that sends a point in the fiber over x to the fiber over $f(x)$). In the rest of these notes, we will not need this construction however as we will mainly be interested in derivatives of real-valued functions (see Chapter 13).

We have also mentioned already in the introduction to this chapter that one often wants to consider functions on a manifold X that assign to any point $x \in X$ a vector in the fiber E_x of a vector bundle $\pi: E \to X$, such as e. g. tangent vector fields. Such functions are called sections of the vector bundle, and they are closely related to morphisms of vector bundles as in the following construction.

Construction 12.14 (Sections of vector bundles)**.** For a vector bundle $\pi: E \to X$, a smooth map $s: X \to E$ with $\pi \circ s = \mathrm{id}_X$ is called a **section** of E; we can think of it as a smooth assignment of a vector in the fiber E_x for any point $x \in X$. Hence, for example, the picture on the right shows a section of the Möbius strip vector bundle of Example 12.1 (where the image of s is the bold curve in E), and a section of the tangent bundle T_X can be thought of as a tangent vector field as in the Hairy Ball Theorem of Proposition 9.5. For any submanifold Y of X, the set of all sections of the restriction $E|_Y$ of Construction 12.4 (a) will be denoted by $E(Y)$. As in Construction 11.15, the **support** of a section $s \in E(Y)$ is defined to be the closure in Y of the set of all $x \in Y$ with $s(x) \neq 0$ in E_x.



There are a few important properties of sections that follow directly from the definitions:

(a) Any section $s: X \to E$ determines an isomorphism between X and its image $s(X)$: It is clear that the morphism $s: X \to s(X)$ is bijective, and $\pi|_{s(X)} : s(X) \to X$ is an inverse morphism.

In particular, there is always the *zero section* sending each point $x \in X$ to the origin in E_x, which allows us to consider X as being naturally embedded in E – given e. g. by the center line of the Möbius strip in the picture above.

(b) A section $s: X \to E$ is the same as a morphism of vector bundles $f: X \times \mathbb{R} \to E$ from the trivial rank-1 vector bundle on X to E: Given a section $s: X \to E$, there is an associated morphism of vector bundles $f: X \times \mathbb{R} \to E$, $(x,\lambda) \mapsto \lambda s(x)$ (with the scalar multiplication in the fiber E_x), and for any morphism $f: X \times \mathbb{R} \to E$ of vector bundles we can construct a section $s = f(\cdot,1): X \to E$. It is clear that these two constructions are inverse to each other. In particular, by Construction 12.6 (a) the spaces of sections $E(Y)$ for any submanifold $Y \subset X$ are vector spaces by pointwise addition and scalar multiplication in the fibers.

(c) A vector bundle $\pi: E \to X$ of rank 1 is isomorphic to a trivial bundle if and only if it has a section which is nowhere zero: Of course, any trivial bundle $\pi: X \times \mathbb{R}^r \to X$ of rank $r > 0$ has a section which is nowhere zero by taking $s: x \mapsto (x, v)$ for an arbitrary constant vector $v \in \mathbb{R}^r \backslash \{0\}$. Conversely, if E has rank 1 and $s: X \to E$ is nowhere zero, then the corresponding morphism of vector bundles $f: X \times \mathbb{R} \to E$ from (b) is bijective, hence an isomorphism by Construction 12.6 (c).

As an example, it seems intuitive that the rank-1 Möbius strip vector bundle of the picture above is non-trivial, and that in fact any section of it must cross the middle line of the strip. In order to be able to phrase this rigorously, let us now give an explicit construction of this vector bundle; it is a special case of the so-called *tautological bundle* of projective spaces.

Construction 12.15 (Tautological subbundles of projective spaces)**.** For $n \in \mathbb{N}$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, consider the subset

$$E = \{((x_0 : \cdots : x_n), (v_0, \ldots, v_n)) \in \mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}^{n+1} : (x_0, \ldots, x_n) \text{ and } (v_0, \ldots, v_n) \text{ are linearly dependent}\}$$

of $\mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}^{n+1}$ together with the projection $\pi \colon E \to \mathbb{P}^n_{\mathbb{K}}$ onto its first component. Note that on the open subset $U_0 = \{(x_0 : \cdots : x_n) : x_0 \neq 0\}$ of $\mathbb{P}^n_{\mathbb{K}}$ with coordinates $\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}$, it is the image subbundle for the matrix-valued function

$$A \colon U_0 \to \mathbb{K}^{(n+1)\times 1},\ (x_0 : \cdots : x_n) \mapsto \left(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)^{\mathsf{T}}. \qquad (*)$$

The same holds over the other open subsets $U_i = \{(x_0 : \cdots : x_n) : x_i \neq 0\}$, so E is a subbundle of the trivial bundle $\mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}^{n+1}$ on $\mathbb{P}^n_{\mathbb{K}}$ by Lemma 12.8 and Remark 12.9 (a). By construction, its fiber over a point $(x_0 : \cdots : x_n) \in \mathbb{P}^n_{\mathbb{K}}$ (which by definition is a 1-dimensional subspace of $\mathbb{K}^{n+1}$) is precisely this 1-dimensional subspace; hence it is called the **tautological subbundle** of $\mathbb{P}^n_{\mathbb{K}}$. As its fibers are all isomorphic to $\mathbb{K}$, it has rank 1 if $\mathbb{K} = \mathbb{R}$ and rank 2 if $\mathbb{K} = \mathbb{C}$.

In fact, we claim that the tautological subbundle of $\mathbb{P}^1_{\mathbb{R}} \cong S^1$ is just the Möbius strip bundle from the introduction to this chapter. To see this, consider first the open subset $U_0 \cong \mathbb{R}$ (on which E is trivial by Lemma 12.8), with the nowhere-zero section obtained by normalizing $(*)$, i. e.

$$s \colon U_0 \to U_0 \times \mathbb{R}^2,\ x \mapsto \left(x, \frac{A(x)}{\|A(x)\|}\right) = \left(x, \frac{(x_0, x_1)^{\mathsf{T}}}{\sqrt{x_0^2 + x_1^2}}\right).$$

Now we go to the point at infinity $\infty = (0:1) \in \mathbb{P}^1_{\mathbb{R}} \backslash U_0$, where correspondingly the fiber of E is $\operatorname{Lin}((0,1)^{\mathsf{T}})$. We can approach this point from both sides by letting $\frac{x_1}{x_0}$ go to $\pm\infty$, obtaining

$$\lim_{x_1/x_0 \to \infty} s(x) = ((0:1), (0,1)^{\mathsf{T}}) \quad \text{and} \quad \lim_{x_1/x_0 \to -\infty} s(x) = ((0:1), (0,-1)^{\mathsf{T}}).$$

Hence, the two ends of the strip over U_0 are glued at this point at infinity in opposite directions, meaning that E over $\mathbb{P}^1_{\mathbb{R}}$ is topologically the Möbius strip (and that $s \colon U_0 \to U_0 \times \mathbb{R}^2$ does not extend to a smooth section of E over $\mathbb{P}^1_{\mathbb{R}}$).

The last example of a vector bundle that we want to consider in this chapter is the *normal bundle* of a submanifold Y of a manifold X. It is a subbundle of $Y \times \mathbb{R}^n$ whose fiber at a point $x \in Y$ is the space of all tangent vectors in $T_{X,x}$ orthogonal to $T_{Y,x}$, and it can be obtained analogously to Construction 12.13.

Construction 12.16 (Normal bundles)**.** Let Y be an r-dimensional submanifold of a k-dimensional manifold $X \subset \mathbb{R}^n$ (again we will see how to avoid a choice of embedding in $\mathbb{R}^n$ in Remark 12.29). At any point $x \in Y$, we define the **normal space** of Y in X at x to be

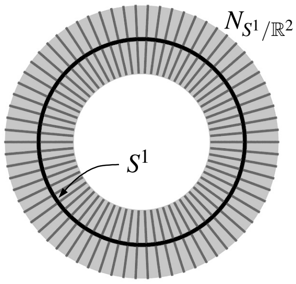
$$N_{Y/X,x} := \{v \in T_{X,x} \subset \mathbb{R}^n : v \perp T_{Y,x}\}, \quad \text{and set} \quad N_{Y/X} := \bigcup_{x \in Y} (\{x\} \times N_{Y/X,x}) \quad \subset Y \times \mathbb{R}^n.$$

To see that $N_{Y/X}$ is a vector bundle on Y, note first by Definition 11.1 that around any $x \in Y$ we have an adapted chart $\varphi \colon U \to V \subset \mathbb{R}^k$ in which Y is given by the vanishing of the last $k - r$ coordinates. Hence, if we form the Jacobian matrix $A_\varphi(x) := (\varphi^{-1})'(\varphi(x)) \in \mathbb{R}^{n \times k}$ exactly as in Construction 12.13, its first r columns will span $T_{Y,x}$, whereas all its columns span $T_{X,x}$.

Now recall the Gram-Schmidt orthonormalization process from linear algebra [G1, Remark 21.27 (c) and Proposition 21.31]: Given any k linearly independent vectors, it exhibits an explicit algorithm (in terms of smooth functions) to add to each of the vectors a certain linear combination of the previous ones so that the new vectors are orthonormal. In terms of matrices, this means that we can write $A_\varphi(x) = Q_\varphi(x) R_\varphi(x)$ where $Q_\varphi(x) \in \mathbb{R}^{n \times k}$ and $R_\varphi(x) \in \mathbb{R}^{k \times k}$ vary smoothly with x, the matrix

$Q_\varphi(x)$ has orthogonal columns, and $R_\varphi(x)$ is upper triangular and invertible. As the first r resp. all columns of $Q_\varphi(x)$ still span $T_{Y,x}$ resp. $T_{X,x}$, this means that in the given chart $N_{Y/X}$ is just the image subbundle $\operatorname{im}\tilde{Q}_\varphi$, where $\tilde{Q}_\varphi(x) \in \mathbb{R}^{n\times(k-r)}$ consists of the last $k-r$ columns of $Q_\varphi(x)$. Hence, by Lemma 12.8 and Remark 12.9 (a) we see that $N_{Y/X}$ is a vector bundle of rank $k-r$ on Y. It is called the **normal bundle** of Y in X.

Example 12.17. For the sphere $Y = S^n$ in $X = \mathbb{R}^{n+1}$ with $n \in \mathbb{N}$, the normal bundle $N_{S^n/\mathbb{R}^{n+1}}$ is a vector bundle of rank 1 on S^n. The picture on the right shows this situation in the case $n = 1$, where as before we have drawn the radial fibers $N_{S^1/\mathbb{R}^2,x}$ as bounded and attached to the corresponding point $x \in S^1$ (which gives a diffeomorphic manifold). Note that $N_{S^n/\mathbb{R}^{n+1}}$ has a nowhere vanishing section by mapping any point $x \in S^n$ to itself in the fiber $\mathbb{R}^{n+1}$, so the normal bundle $N_{S^n/\mathbb{R}^{n+1}}$ is in fact (isomorphic to) a trivial bundle by Construction 12.14 (c), although its embedding in $S^n \times \mathbb{R}^{n+1}$ is non-trivial.



By construction, note that we have $T_{Y,x} \oplus N_{Y/X,x} = T_{X,x}$ for any point $x \in Y$ if $Y \subset X \subset \mathbb{R}^n$ are submanifolds. As expected, we can make this into an equation of vector bundles by taking this equation fiber-by-fiber:

Construction 12.18 (Direct sums of subbundles)**.** Let F_1 and F_2 be two subbundles of a vector bundle $\pi\colon E \to X$, and assume that their fibers $F_{1,x}$ and $F_{2,x}$ form a direct sum in E_x for all $x \in X$, i. e. that $F_{1,x} \cap F_{2,x} = \{0\}$. Then we can form their fiberwise direct sum

$$F_1 \oplus F_2 := \bigcup_{x\in X} (F_{1,x} \oplus F_{2,x}),$$

which is again a vector subbundle of E on X, of rank $\operatorname{rk} F_1 + \operatorname{rk} F_2$: By Remark 12.9 (b) we may assume that F_1 and F_2 are image subbundles given by two matrix-valued functions with linearly independent columns at every point, and $F_1 \oplus F_2$ is then just the image subbundle given by joining these columns into one matrix.

Example 12.19.

(a) By construction, we have $T_Y \oplus N_{Y/X} = T_X|_Y$ as bundles on Y for any submanifold Y of a manifold $X \subset \mathbb{R}^n$.

(b) As a concrete case, consider $Y = S^2$ in $X = \mathbb{R}^3$, and note that $T_X|_Y = Y \times \mathbb{R}^3$ is trivial. Hence, by (a) we have

$$T_{S^2} \oplus N_{S^2/\mathbb{R}^3} = S^2 \times \mathbb{R}^3.$$

But $N_{S^2/\mathbb{R}^3}$ is trivial as well by Example 12.17, whereas T_{S^2} is not: By the Hairy Ball Theorem of Proposition 9.5, it does not have a nowhere vanishing section, and thus cannot be trivial by Construction 12.14 (c). Hence, a direct sum of a non-trivial and a trivial bundle can in fact be trivial – which might be counterintuitive, but it is of course the non-trivial embedding of $N_{S^2/\mathbb{R}^3}$ in $S^2 \times \mathbb{R}^3$ that makes this work.

Exercise 12.20.

(a) Show that the tangent bundle of the sphere S^1 is trivial.

(b) Note that the tangent bundle of S^2 is not trivial by the Hairy Ball Theorem of Proposition 9.5.

Is the tangent bundle of S^3 trivial?

An important application of the normal bundle $N_{Y/X}$ of a submanifold Y of a manifold X can already be seen from the picture in Example 12.17: Considered as a manifold itself, it seems that we can think of $N_{Y/X}$ as a homotopy equivalent open neighborhood of Y in X. Let us prove this statement now, which is usually called the *Tubular Neighborhood Theorem* since in the case $\dim Y = 1$ and

$\dim X = 3$ the (bounded) normal bundle $N_{Y/X}$ has 2-dimensional disks as fibers and thus looks like a tube around the 1-dimensional manifold Y.

Proposition 12.21 (Tubular Neighborhood Theorem). *For any compact submanifold X of $\mathbb{R}^n$, there is an $\varepsilon > 0$ such that the ε-neighborhood of the zero section of the normal bundle*

$$N^{\varepsilon}_{X/\mathbb{R}^n} := \{(x,v) \in N_{X/\mathbb{R}^n} : \|v\| < \varepsilon\} \quad \subset X \times \mathbb{R}^n$$

is mapped isomorphically by

$$f\colon N^{\varepsilon}_{X/\mathbb{R}^n} \to \mathbb{R}^n,\ (x,v) \mapsto x+v$$

21 *onto an open neighborhood of X in $\mathbb{R}^n$.*

Proof. Let us first consider the map $f\colon N_{X/\mathbb{R}^n} \to \mathbb{R}^n$, $(x,v) \mapsto x+v$ on the whole normal bundle and show that it is a local isomorphism around any point $(x,0)$ on the zero section. This is just a straightforward application of the inverse function theorem of Remark 11.4: Setting $k = \dim X$, consider a chart $\varphi\colon U \to V \subset \mathbb{R}^k$ for X around x and write $N_{X/\mathbb{R}^n}$ on this chart as an image subbundle $\operatorname{im}\tilde{Q}_\varphi$ with $\tilde{Q}_\varphi(x) \in \mathbb{R}^{n\times(n-k)}$ a matrix with orthonormal columns obtained by Gram-Schmidt normalization as in Construction 12.16. In the coordinates $(y,u) \in V \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$ of the chart φ and the local trivialization of $N_{X/\mathbb{R}^n}$, the map f is then given by

$$\tilde{f}\colon V \times \mathbb{R}^{n-k} \to \mathbb{R}^n,\ (y,u) \mapsto \varphi^{-1}(y) + \tilde{Q}_\varphi(\varphi^{-1}(y)) \cdot u.$$

Its Jacobian matrix at a point $(y,0)$ is $\big((\varphi^{-1})'(y)\,\big|\,\tilde{Q}_\varphi(\varphi^{-1}(y))\big) \in \mathbb{R}^{n\times n}$, which is invertible since the first k columns span the tangent space and the last $n-k$ columns the normal space of X at that point. Hence, the map f is a local isomorphism around any point on the zero section of $N_{X/\mathbb{R}^n}$.

Hence, for any $x \in X$ we can choose an open neighborhood $U_x \subset X$ of x and $\varepsilon_x > 0$ such that f is a local isomorphism on $N^{\varepsilon_x}_{U_x/\mathbb{R}^n}$. But X is compact, so finitely many $U_{x_1},\dots,U_{x_r}$ suffice to cover X. Choosing any $\varepsilon < \min(\varepsilon_{x_1},\dots,\varepsilon_{x_r})$, the map f is thus an isomorphism from $N^{\varepsilon}_{X/\mathbb{R}^n}$ onto an open neighborhood of X.

It just remains to be shown that f is also a global isomorphism onto its image, i. e. that it is injective. We will do this by contradiction: Assume that this was false for any ε, then we can choose sequences $(x_i,v_i)_{i\in\mathbb{N}_{>0}}$ and $(\tilde{x}_i,\tilde{v}_i)_{i\in\mathbb{N}_{>0}}$ with $f(x_i,v_i) = f(\tilde{x}_i,\tilde{v}_i)$, i. e. $x_i + v_i = \tilde{x}_i + \tilde{v}_i$, and $\|v_i\| < \frac{1}{i}$ as well as $\|\tilde{v}_i\| < \frac{1}{i}$ for all i. For $i \to \infty$ we clearly have $v_i \to 0$ and $\tilde{v}_i \to 0$, and as X is compact we may assume after taking a subsequence that $x_i \to x$ and $\tilde{x}_i \to \tilde{x}$ converge in X. But since $x+v = \tilde{x}+\tilde{v}$, this requires $x = \tilde{x}$. Hence, the map f is actually also locally not injective around $(x,0)$, which is a contradiction to our construction. $\square$

Remark 12.22.

(a) Note that the map f of the proof of Proposition 12.21 is just what we did in the picture of Example 12.17: We attached the normal spaces to their base point, i. e. moved $N_{X/\mathbb{R}^n,x}$ so that its origin is at $x \in X$.

(b) Rescaling the fibers as in Remark 10.10 (b), we see that in fact the normal bundle $N_{X/\mathbb{R}^n}$ itself is isomorphic to an open neighborhood of X in $\mathbb{R}^n$. Moreover, by Remark 12.5 (a) the submanifold X is a deformation retract of this neighborhood and thus has the same homology. We can thus think of the normal bundle $N_{X/\mathbb{R}^n}$ as a manifold analogue of the simplicial standard neighborhoods of Proposition 1.24.

(c) As you might expect, there are more general versions of the Tubular Neighborhood Theorem:

- The analogous statement also holds for a compact submanifold Y of a manifold $X \subset \mathbb{R}^n$, i.e. Y has a open neighborhood in X that is isomorphic to the normal bundle $N_{Y/X}$. The main change in the proof is then needed since the map $f\colon N_{Y/X} \to \mathbb{R}^n$, $(x,v) \mapsto x+v$ does not map to X, but only to $\mathbb{R}^n$. The solution to this problem is to use the Tubular Neighborhood Theorem for $X/\mathbb{R}^n$ as well to locally identify $\mathbb{R}^n$ with the normal bundle $N_{X/\mathbb{R}^n}$, hence we can consider f as locally mapping to $N_{X/\mathbb{R}^n}$, compose it with the bundle projection $\pi\colon N_{X/\mathbb{R}^n} \to X$ to map to X, and then apply an analogous proof as above to the map $\pi \circ f: N_{Y/X} \to X$ [B, Theorem II.11.14].

- Actually, there is also way to state and prove the theorem without using any embeddings in $\mathbb{R}^n$; we will briefly discuss this in Remark 12.29.
- The submanifold X does not have to be compact; in this case however one cannot choose a global ε but needs a neighborhood of the zero section in the normal bundle whose size varies with the point on the base.

Let us finish this chapter with an alternative construction of vector bundles. Note that all bundles on a manifold X that we have considered so far were subbundles of a trivial bundle $X \times \mathbb{R}^n$ for some $n \in \mathbb{N}$ – but this is clearly not required by definition, similarly to manifolds that are not required a priori to be submanifolds of some $\mathbb{R}^n$. In particular, considering only subbundles of $X \times \mathbb{R}^n$ allowed us to construct tangent and normal bundles only for submanifolds of $\mathbb{R}^n$ so far. The following construction using transition maps overcomes this problem and allows us to obtain more general vector bundles by a gluing process.

Construction 12.23 (Vector bundles from transition maps)**.**

(a) Let $\pi: E \to X$ be a vector bundle of rank r, and let $(U_i)_{i \in I}$ be an open cover of X over which π is trivial, i. e. such that there are local trivialization isomorphisms

$$\varphi_i: \pi^{-1}(U_i) \to U_i \times \mathbb{R}^r$$

over U_i for all $i \in I$. Similarly to manifolds, for any $i, j \in I$ there are then **transition maps**

$$\varphi_j \circ \varphi_i^{-1}: (U_i \cap U_j) \times \mathbb{R}^r \to (U_i \cap U_j) \times \mathbb{R}^r$$

which are isomorphisms of trivial vector bundles over $U_i \cap U_j$, and thus by Construction 12.6 must be of the form

$$\varphi_j \circ \varphi_i^{-1}: (U_i \cap U_j) \times \mathbb{R}^r \to (U_i \cap U_j) \times \mathbb{R}^r, \; (x, v) \mapsto (x, A_{i,j}(x) \cdot v)$$

for smooth matrix-valued maps $A_{i,j}: U_i \cap U_j \to \mathrm{GL}(r, \mathbb{R})$, that by abuse of notation are also sometimes called the *transition maps* of the bundle. Note that for any $i, j, k \in I$ we clearly have $\varphi_i \circ \varphi_i^{-1} = \mathrm{id}$ and $\varphi_k \circ \varphi_i^{-1} = (\varphi_k \circ \varphi_j^{-1}) \circ (\varphi_j \circ \varphi_i^{-1})$ where defined, and thus there are compatibility relations between these maps

$$A_{i,i} = E \quad \text{and} \quad A_{i,k} = A_{j,k} \cdot A_{i,j}: U_i \cap U_j \cap U_k \to \mathrm{GL}(r, \mathbb{R}). \tag{$*$}$$

(b) The construction of (a) can actually be reversed: Let $(U_i)_{i \in I}$ be an open cover of a manifold X, and assume that for some $r \in \mathbb{N}$ we are given smooth maps $A_{i,j}: U_i \cap U_j \to \mathrm{GL}(r, \mathbb{R})$ for all $i, j \in I$ satisfying the compatibility relations $(*)$ for all $i, j, k \in I$. Then we can construct from these data a vector bundle E of rank r on X having these transition maps by taking the disjoint union of $U_i \times \mathbb{R}^r$ for all $i \in I$ modulo the equivalence relation

$$(x, v) \in U_i \times \mathbb{R}^r \quad \sim \quad (\tilde{x}, \tilde{v}) \in U_j \times \mathbb{R} \qquad :\Leftrightarrow \qquad x = \tilde{x} \text{ and } \tilde{v} = A_{i,j}(x) \cdot v,$$

giving the resulting space the quotient topology (note that this is actually an equivalence relation by $(*)$).

Exercise 12.24. Let $(U_i)_{i \in I}$ be an open cover of a manifold X, and let $A_{i,j}, B_{i,j}: U_i \cap U_j \to \mathrm{GL}(r, \mathbb{R})$ for $i, j \in I$ be two sets of transition maps for vector bundles, i. e. such that

$$A_{i,i} = B_{i,i} = E, \quad A_{i,k} = A_{j,k} \cdot A_{i,j}, \quad B_{i,k} = B_{j,k} \cdot B_{i,j}$$

(where defined) for all $i, j, k \in I$. Show that the vector bundle with transition maps $A_{i,j}$ as in Construction 12.23 (b) is isomorphic to the vector bundle with transition maps $B_{i,j}$ if and only if there exist smooth maps $\Lambda_i: U_i \to \mathrm{GL}(r, \mathbb{R})$ with

$$A_{i,j} = \Lambda_j \cdot B_{i,j} \cdot \Lambda_i^{-1} \quad \text{on } U_i \cap U_j$$

for all $i, j \in I$.

Example 12.25 (Tangent bundles revisited)**.** Let X be a k-dimensional manifold.

(a) Assume first that X is embedded as a submanifold of $\mathbb{R}^n$ for some n, with smooth structure $(\varphi_i)_{i\in I}$ of X for $\varphi_i\colon U_i \to V_i \subset \mathbb{R}^k$. Recall by Lemma 12.8 and Construction 12.13 (a) that on U_i a local trivialization ψ_i of the tangent bundle $\pi\colon T_X \to X$ is given by

$$\psi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k,\ (x, (\varphi_i^{-1})'(\varphi_i(x))\cdot v) \mapsto (x,v).$$

Forming the composition $\psi_j \circ \psi_i^{-1}$ on $(U_i \cap U_j) \times \mathbb{R}^k$ for another $j \in I$, we thus first send (x,v) by ψ_i^{-1} to

$$(x, (\varphi_i^{-1})'(\varphi_i(x))\cdot v) = (x, (\varphi_j^{-1})'(\varphi_j(x))\cdot(\varphi_j\circ\varphi_i^{-1})'(\varphi_i(x))\cdot v),$$

which is then sent by ψ_j to $(x, (\varphi_j\circ\varphi_i^{-1})'(\varphi_i(x))\cdot v)$. Hence, the transition maps for the tangent bundle are given by the matrix-valued functions

$$A_{i,j}\colon U_i\cap U_j \to \mathrm{GL}(k,\mathbb{R}),\ x\mapsto A_{i,j}(x) = (\varphi_j\circ\varphi_i^{-1})'(\varphi_i(x)), \qquad (*)$$

i. e. by the Jacobian matrices of the transition maps of the manifold.

(b) Using Construction 12.23 (b), we can now define the **tangent bundle** T_X of an arbitrary manifold X with smooth structure $(\varphi_i)_{i\in I}$ so that it coincides with our old definition by (a): It is simply the vector bundle of rank k on X given by the transition maps $(*)$. Note that if X does not have a natural embedding in $\mathbb{R}^n$ then T_X does not have a natural embedding in $X\times\mathbb{R}^n$ either, but it is still a natural vector bundle on X that can be defined without making any choices and has the usual geometric interpretation as collecting local linearizations of X.

Exercise 12.26. Let $f\colon X\to Y$ be a morphism of manifolds such that X is connected and $f'(x)\colon T_{X,x}\to T_{Y,f(x)}$ is the zero map for all $x\in X$. Show that f is a constant map.

Exercise 12.27. Let $X\subset\mathbb{R}^n$ be a submanifold.

(a) Show that $\Delta := \{(x,x) : x\in X\}$ is a submanifold of $X\times X$, and that

$$\delta\colon X\to\Delta,\ x\mapsto(x,x)$$

is an isomorphism.

(b) Show that the pullback bundle $\delta^* N_{\Delta/X\times X}$ as in Exercise 12.11 is isomorphic to the tangent bundle T_X.

Exercise 12.28. In this exercise we want to see an alternative construction of tangent spaces and tangent bundles that does not use embeddings in $\mathbb{R}^N$. Let X be an n-dimensional manifold, and let $\varphi\colon U\to V\subset\mathbb{R}^n$ be a chart. Recall that $\Omega^0(U)$ denotes the vector space of smooth real-valued functions on U.

(a) Show that the maps

$$\frac{\partial}{\partial\varphi_i}\colon \Omega^0(U)\to\Omega^0(U),\ f\mapsto\frac{\partial f}{\partial\varphi_i} := \frac{\partial(f\circ\varphi^{-1})}{\partial x_i}\circ\varphi$$

for $1\le i\le n$ define linear homomorphisms verifying the Leibniz rule, i. e. such that

$$\frac{\partial(fg)}{\partial\varphi_i} = \frac{\partial f}{\partial\varphi_i}\cdot g + f\cdot\frac{\partial g}{\partial\varphi_i} \quad \text{for all } f,g\in\Omega^0(U).$$

(b) For all $x\in U$ and $i\in\{1,\dots,n\}$, consider the linear map

$$\left.\frac{\partial}{\partial\varphi_i}\right|_x\colon \Omega^0(X)\to\mathbb{R},\ f\mapsto\frac{\partial f|_U}{\partial\varphi_i}(x).$$

Show that $\left(\left.\frac{\partial}{\partial\varphi_i}\right|_x\right)_{1\le i\le n}$ is a basis of the vector space D_x of all linear maps $v\colon\Omega^0(X)\to\mathbb{R}$ satisfying $v(fg) = v(f)\cdot g(x) + f(x)\cdot v(g)$ for all $f,g\in\Omega^0(X)$.

(c) Show that the vector spaces D_x for $x\in X$ form a vector bundle over X isomorphic to the tangent bundle T_X.

Remark 12.29 (Outlook on Riemannian Manifolds)**.** In this chapter, we have often considered manifolds X embedded in $\mathbb{R}^n$, or vector bundles on a manifold X embedded as subbundles in the trivial bundle $X \times \mathbb{R}^n$.

Sometimes, the reason for this was just convenience: For example, we have a geometric intuition what tangent spaces to a submanifold X of $\mathbb{R}^n$ should be, and thus we initially defined the tangent bundle in this way in Construction 12.13. However, we now saw in Example 12.25 that this assumption of an embedding is not essential; we can define the tangent bundle of a manifold just as easily without it using transition maps.

In some of our constructions however the situation was more subtle: If Y is a submanifold of X, to define the normal subspace $N_{Y/X,x}$ of $T_{X,x}$ in Construction 12.16 we needed a notion of tangent vectors being orthogonal – which we obtained from the standard scalar product on $\mathbb{R}^n$ as the assumption $X \subset \mathbb{R}^n$ ensured that $T_{X,x} \subset \mathbb{R}^n$ as well. In the Tubular Neighborhood Theorem of Proposition 12.21 we used the metric induced by this scalar product again to restrict the normal bundle to a well-defined ε-neighborhood of its zero section, and more importantly to construct the map f that "adds a small normal vector to a point in the submanifold" to obtain a point in the ambient manifold – which only works well if the ambient manifold is $\mathbb{R}^n$. All these constructions relied essentially on our manifolds (or at least the fibers of our vector bundles) to live in a globally fixed $\mathbb{R}^n$; there is no analogous construction for arbitrary manifolds and vector bundles without such an embedding.

Actually, this is not quite true: While it is clearly correct that the constructions mentioned above do not work for arbitrary manifolds, we do not really need an embedding in $\mathbb{R}^n$ for them, but just some notion of scalar product resp. metric. This is provided by a so-called *Riemannian manifold*, which is defined to be a smooth manifold X together with a scalar product on each tangent space that varies smoothly with the point (in a sense that we could easily make precise with our current methods: Every point $a \in X$ must have a neighborhood U on which T_X is trivial, i. e. isomorphic to $U \times \mathbb{R}^k$ with $k = \dim X$, such that the scalar product on the tangent spaces over U is given in this local trivialization by a smooth function $U \to \mathbb{R}^{k\times k}$ into the space of all positive definite symmetric matrices). A smooth manifold embedded in $\mathbb{R}^n$ just inherits this scalar product from its ambient space, hence it is automatically a Riemannian manifold.

By construction, a Riemannian manifold enables us to define lengths and angles for tangent vectors at every point, in particular orthogonality. But in fact it allows much more: Recall from analysis that the length of a (smooth) path in $\mathbb{R}^n$ can be computed as the integral over the lengths of its tangent vectors, hence we can also define this notion on a Riemannian manifold. This makes every Riemannian manifold into a metric space by taking the shortest-path metric. Such shortest paths (also called geodesics) are roughly equivalent to straight lines in $\mathbb{R}^n$, and they allow us to define e. g. what it means to start from a point and move "in a straight way" into a certain tangent or normal direction for a certain length – which is needed for the map f in the Tubular Neighborhood Theorem of Proposition 12.21. Moreover, it can be seen that Riemannian manifolds also admit to measure the size of higher-dimensional submanifolds, to talk about curvature, and many other things.

The study of Riemannian manifolds is part of a field of mathematics called *differential geometry*. As you might expect, it would be easy to fill at least a one-semester course with only this. In these notes however we do not want to consider this topic any further as our goal is the *topological* and not the *metric* study of manifolds. For us, for example, the plane $\mathbb{R}^2$, an open disk in $\mathbb{R}^2$ of any size, an open ellipse in $\mathbb{R}^2$, and the open upper half sphere of S^2 are all the same smooth manifold, but they all have different size and curvature properties and thus are completely different Riemannian manifolds.

However, some statements on manifolds are purely topological, yet their most natural proof is done by making them into Riemannian manifolds in an arbitrary way (e. g. by using a Whitney embedding as in Proposition 11.21 or Remark 11.22 (b)) and applying techniques from differential geometry. Our current example for this is the Tubular Neighborhood Theorem as in Remark 12.22 (c): If Y is any submanifold of a manifold X, not embedded in any $\mathbb{R}^n$, one can in fact define its normal bundle $N_{Y/X}$ abstractly (i. e. without embedding it in $T_X|_Y$ or a trivial bundle and without using orthogonality) as a quotient bundle $T_X|_Y/T_Y$, making the varying quotient vector spaces $T_{X,x}/T_{Y,x}$

for $x \in Y$ into a vector bundle on Y. It is then a purely topological statement that this bundle is isomorphic to an open neighborhood of Y in X – but to prove it one has to use metric techniques as we did in Proposition 12.21.