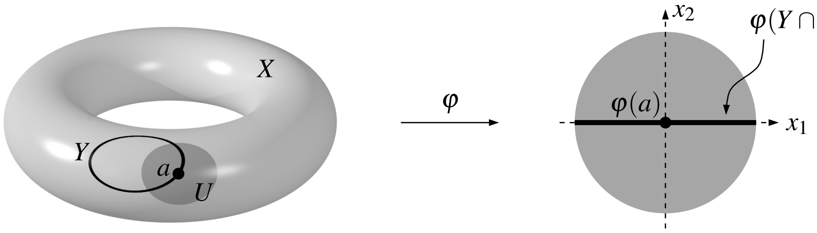
11. Submanifolds

Similarly to most other categories, having defined manifolds we should now study *submanifolds*, i. e. subsets of a manifold that are again manifolds in a natural way. Of course, one obvious case here is any open subset of a manifold, which we have seen in Example 10.17 (b) to be a manifold as well. But this situation is very special since both manifolds have the same dimension – one could clearly also imagine cases in which a manifold contains a smaller-dimensional one, such as the torus X containing the circle Y in the picture below on the left. Another example that you might know already from first-year mathematics is the case of submanifolds of $\mathbb{R}^n$ [G1, Definition 27.18]: Drawing the torus embedded in $\mathbb{R}^3$ as below means that we have exhibited it as a submanifold of $\mathbb{R}^3$.



The idea to define such submanifolds is very similar to the definition of manifolds themselves: Just as an n-dimensional manifold is a space that locally looks like $\mathbb{R}^n$, a k-dimensional submanifold of it is a space that locally looks like the standard embedding of $\mathbb{R}^k$ in $\mathbb{R}^n$ setting the last $n-k$ coordinates to zero. In the picture above, this means that the circle Y in the torus X is locally given by the equation $x_2 = 0$ in a suitable chart φ. Let us make this definition precise:

Definition 11.1 (Submanifolds)**.** For $k \leq n$, a non-empty subset Y of an n-dimensional manifold X is called a k-dimensional **submanifold** of X if for any point $a \in Y$ there is a chart $\varphi\colon U \to V \subset \mathbb{R}^n$ for X with $a \in U$ such that $\varphi(U \cap Y) = \{x \in V : x_{k+1} = \cdots = x_n = 0\}$. Such a chart will be called **adapted to** Y. Clearly, by a translation in $\mathbb{R}^n$ we might assume that $\varphi(a) = 0$ if we wish.

Of course, the first thing to check is that submanifolds in this sense are actually manifolds themselves. Although this is expected from the geometric intuition, it is not obvious from the definition.

Construction 11.2 (Submanifolds as manifolds)**.** Let Y be a k-dimensional submanifold of an n-dimensional manifold X. We will construct from this a natural k-dimensional atlas on Y by restricting adapted charts on X to Y.

More precisely, for any $a \in Y$ let $\varphi\colon U \to V$ be a chart on X adapted to Y with $a \in U$. By restricting φ to Y in the source and to the first k coordinates in the target, this defines a chart

$$\tilde{\varphi}\colon \tilde{U} \to \tilde{V},\ x \mapsto (\varphi_1(x), \ldots, \varphi_k(x))$$

containing a on Y, where $\tilde{U} = U \cap Y$ is open in Y, $\tilde{V} = \{(x_1,\ldots,x_k) \in \mathbb{R}^k : (x_1,\ldots,x_k,0,\ldots,0) \in V\}$ is open in $\mathbb{R}^k$, and $\varphi_1,\ldots,\varphi_k$ are first k component functions of φ.

Note that these charts are compatible: For any two charts φ_1, φ_2 on X the transition map $\tilde{\varphi}_2 \circ \tilde{\varphi}_1^{-1}$ is just a restriction of $\varphi_2 \circ \varphi_1^{-1}$, and thus smooth. Hence the charts $\tilde{\varphi}$ for all adapted charts φ form an atlas on Y. As the second countability and Hausdorff conditions pass to arbitrary subsets, this makes Y into a k-dimensional manifold.

Moreover, this manifold structure on Y has the property that it makes the embedding $Y \to X$ into a morphism: Taking an adapted chart φ on X and the corresponding chart $\tilde{\varphi}$ on Y, the inclusion is given by $(x_1,\ldots,x_k) \mapsto (x_1,\ldots,x_k,0,\ldots,0)$, which is a smooth map between open subsets of $\mathbb{R}^k$ and $\mathbb{R}^n$.

We will see examples of submanifolds soon, but to simplify their study let us first discuss some general methods to construct them. Thinking of other categories, one would probably expect that new submanifolds can be obtained from old ones by taking inverse images or images under morphisms, so let us investigate this strategy. Unfortunately, without any further restrictions this fails badly, as the following examples show.

Example 11.3.

(a) Let $f\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto xy$. Then the origin in $\mathbb{R}$ is clearly a 0-dimensional submanifold, but its inverse image $f^{-1}(\{0\}) = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$ is not by Example 10.4 (e). Note that the derivative of f is $f' = (y,x)$, so the rank of f is 0 at the origin, whereas it is 1 everywhere else.

(b) Consider the vertical projection $f\colon S^1 \to \mathbb{R},\ (x,y) \mapsto x$. Then S^1 is clearly a manifold, but its image $f(S^1) = [-1,1] \subset \mathbb{R}$ is not, again by Example 10.4 (e). In this case as well, note by Example 10.7 (b) that f has rank 0 at the points $(\pm 1, 0)$ (which map to the critical boundary points of $[-1,1]$), whereas it has rank 1 everywhere else.

Hence, it seems that a rank condition on a morphism $f\colon X \to Y$ could solve our problems. In fact, we will show now that with such a rank condition one can bring f in suitable coordinates in a *normal form* which plays well with adapted charts. For simplicity, we will do this here only in the cases when the rank of f is maximal in the sense that it is either $\dim Y$ (in which the derivative f' in given charts is surjective at any point) or $\dim X$ (in which f' is injective), leading to good behavior with respect to inverse images or images of submanifolds, respectively. The key input to pass from the derivative f' to an actual local behavior of f is the inverse function theorem, which we briefly recall.

Remark 11.4 (Inverse function theorem)**.** In the following, we will need the *inverse function theorem* from analysis [G1, Proposition 27.6]: Any smooth map $f\colon U \to V$ with $f(0) = 0$ and invertible Jacobian matrix $f'(0)$ between two open neighborhoods U and V of 0 in $\mathbb{R}^n$ is a local diffeomorphism, i.e. after possibly shrinking U and V to smaller neighborhoods of the origin the map f will be bijective with a smooth inverse f^{-1}. Actually, this theorem is usually proven for continuously differentiable instead of smooth maps, but in fact it holds for smooth maps as well: As $(f^{-1})'(x) = (f'(f^{-1}(x)))^{-1}$ [G1, Remark 27.3], any order of differentiability that holds for f' and f^{-1} also holds for $(f^{-1})'$, hence by induction if f is smooth then so is f^{-1}.

Lemma 11.5 (Local normal form of smooth maps)**.** *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open neighborhoods of the origin, and let $f\colon U \to V$ be a smooth map with $f(0) = 0$.*

(a) *If $\operatorname{rk}_0 f = m$ (so in particular $m \le n$), after possibly shrinking U and V there is a diffeomorphism $\varphi\colon U \to W$ (i.e. a coordinate change) to an open neighborhood W of 0 in $\mathbb{R}^n$ such that $\varphi(0) = 0$ and $f \circ \varphi^{-1}\colon W \to V$ is the map $s\colon (x_1,\dots,x_n) \mapsto (x_1,\dots,x_m)$.*

(b) *If $\operatorname{rk}_0 f = n$ (so in particular $n \le m$), after possibly shrinking U and V there is a diffeomorphism $\varphi\colon V \to W$ to an open neighborhood W of 0 in $\mathbb{R}^m$ such that $\varphi(0) = 0$ and $\varphi \circ f\colon U \to W$ is the map $i\colon (x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,0,\dots,0)$.*

Proof.

(a) By a permutation of the coordinates in $\mathbb{R}^n$ (which can be absorbed into φ) we may assume without loss of generality that $A := \frac{\partial f}{\partial(x_1,\dots,x_m)}(0)$ is invertible. We construct the auxiliary function

$$F\colon U \to V \times \mathbb{R}^{n-m},\ x \mapsto (f(x), x_{m+1},\dots,x_n) \quad \text{with} \quad F'(0) = \left(\begin{array}{c|c} A & * \\ \hline 0 & E \end{array}\right).$$

As $F'(0)$ is then invertible as well, the inverse function theorem of Remark 11.4 tells us that F is a local diffeomorphism of $\mathbb{R}^n$ around 0. As $f = s \circ F$, we then have $f \circ \varphi^{-1} = s$ with $\varphi := F$.

(b) This time we can permute the coordinates of $\mathbb{R}^m$ to assume that $A := \frac{\partial(f_1,\dots,f_n)}{\partial x}(0)$ is invertible, where $f_1,\dots,f_m$ are the component functions of f. We now construct the auxiliary function

$$F\colon U\times\mathbb{R}^{m-n}\to\mathbb{R}^m=\mathbb{R}^n\times\mathbb{R}^{m-n},\ (x,y)\mapsto f(x)+(0,y)\quad\text{with}\quad F'(0)=\left(\begin{array}{c|c} A & 0 \\ \hline * & E\end{array}\right).$$

Again, $F'(0)$ is invertible as well, so F is a local diffeomorphism of $\mathbb{R}^m$ around 0 by Remark 11.4. As $f = F\circ i$, we thus obtain $\varphi\circ f = i$ with $\varphi := F^{-1}$. □

Exercise 11.6. As a generalization of Lemma 11.5, let $U\subset\mathbb{R}^n$ and $V\subset\mathbb{R}^m$ again be open neighborhoods of the origin, and let $f\colon U\to V$ be a smooth map with $f(0)=0$ such that $r:=\operatorname{rk}_a f$ is constant for $a\in U$.

(a) Show that after possibly shrinking U and V there are diffeomorphisms $\varphi\colon U\to U'$ and $\psi\colon V\to V'$ to open neighborhoods U' and V' of the origin in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, such that $\psi\circ f\circ\varphi^{-1}\colon U'\to V'$ is the map $(x_1,\dots,x_n)\mapsto(x_1,\dots,x_r,0,\dots,0)$.

(b) Deduce the two parts of Lemma 11.5 directly as special cases of (a).

17

Let us now pass from this local the global setting.

Definition 11.7 (Submersions and immersions)**.** Let $f\colon X\to Y$ be a morphism of manifolds.

(a) The morphism f is called a **submersion** if $\operatorname{rk}_a f=\dim Y$ for all $a\in X$.

(b) The morphism f is called an **immersion** if $\operatorname{rk}_a f=\dim X$ for all $a\in X$.

Example 11.8. By the normal form of Lemma 11.5, any submersion is a locally surjective, and any immersion is locally injective. However, it is important to understand that the conditions of Definition 11.7 are not equivalent to local surjectivity or injectivity (intuitively since they require that surjectivity or injectivity are visible already on linear approximations, i. e. that we have “local linear surjectivity or injectivity”), and that they clearly do not imply global surjectivity or injectivity:

(a) The morphism $f\colon\mathbb{R}\to\mathbb{R}$, $x\mapsto x^3$ of Example 10.17 (c) is globally and thus also locally bijective, but neither a submersion nor an immersion since its rank at the origin is 0.

(b) The inclusion $f\colon U\to X$ of any non-empty proper open subset U in a manifold X is a submersion and an immersion, but not surjective.

(c) The morphism $f\colon\mathbb{R}\to\mathbb{R}^2$, $t\mapsto(\cos t,\sin t)$ that covers the circle $S^1\subset\mathbb{R}^2$ infinitely often by the real line is an immersion, but not injective.

Proposition 11.9 (Inverse images and images of submanifolds)**.** *Let $f\colon X\to Y$ be a morphism of manifolds with $n:=\dim X$ and $m:=\dim Y$.*

(a) *If f is a submersion and Z is a k-dimensional submanifold of Y, then $f^{-1}(Z)$ is a submanifold of X of dimension $k+n-m$ if it is non-empty.*

(b) *If f is an immersion and Z is a k-dimensional submanifold of X such that $f|_Z\colon Z\to f(Z)$ is a homeomorphism onto its image, then $f(Z)$ is a submanifold of Y of dimension k, and this restriction $f|_Z$ is an isomorphism between Z and $f(Z)$.*

Proof.

(a) The submanifold condition is local, so we can restrict f to open neighborhoods $U\subset X$ of a point $a\in f^{-1}(Z)$ and $V\subset Y$ of $f(a)$ in the following way:

- By Remark 10.18 (a) we may assume that $U\subset\mathbb{R}^n$ and $V\subset\mathbb{R}^m$, and that a and $f(a)$ are the origin.
- On the target $V\subset Y$, we can pick adapted coordinates $x_1,\dots,x_m$ as in Definition 11.1 so that Z is given by the equations $x_{k+1}=\cdots=x_m=0$.
- On the source $U\subset X$, we can then pick coordinates as in Lemma 11.5 (a) such that f is given by $(x_1,\dots,x_n)\mapsto(x_1,\dots,x_m)$.

Then $f^{-1}(Z)\cap U$ is given by the equations $x_{k+1}=\cdots=x_m=0$ again (with the coordinates $x_1,\dots,x_k,x_{m+1},\dots,x_n$ being unrestricted), hence it is a submanifold of dimension $k+n-m$.

(b) Now let $a\in Z$. We use the same strategy as in (a) to find open neighborhood charts U of $a=0$ in $\mathbb{R}^n$ and V of $f(a)=0$ in $\mathbb{R}^m$ such that Z is given by $x_{k+1}=\cdots=x_n=0$, and f is given by $(x_1,\dots,x_n)\mapsto(x_1,\dots,x_n,0,\dots,0)$, this time using Lemma 11.5 (b). As $f|_Z$ is a homeomorphism onto its image, the open set $Z\cap U$ in Z is mapped to an open set $f(Z\cap U)$ in $f(Z)$ contained in $f(Z)\cap V$, so by shrinking V we may assume that $f(Z)\cap V=f(Z\cap U)$. Now the proof continues analogously to (a): The set $f(Z)\cap V$ is given by the equations $x_{k+1}=\cdots=x_n=x_{n+1}=\cdots=x_m=0$, so it is a submanifold of dimension k.

Finally, note that the homeomorphism $f|_Z\colon Z\to f(Z)$ is given in these adapted coordinates by $(x_1,\dots,x_k)\mapsto(x_1,\dots,x_k,0,\dots,0)$, so it is clearly an isomorphism onto its image. □

Remark 11.10.

(a) As one can see from the proof, the statement of Proposition 11.9 also holds under the weaker assumption that the rank of f is equal to $\dim Y$ only at all points of $f^{-1}(Z)$ in part (a), and equal to $\dim X$ only at all points of Z in part (b).

(b) Recall that any bijective continuous map from a compact space to a Hausdorff space is already a homeomorphism [G3, Corollary 4.14], so for a compact submanifold $Z\subset X$ the homeomorphism assumption for $f|_Z$ in Proposition 11.9 (b) can be weakened to injectivity. Without the compactness condition this is not possible however, as we will see in Example 11.11 (c).

Example 11.11.

(a) The most common example of Proposition 11.9 (a) is clearly a submersion $f\colon\mathbb{R}^n\to\mathbb{R}^k$ and the 0-dimensional submanifold $Z=\{0\}$ of $\mathbb{R}^k$, leading to an $(n-k)$-dimensional submanifold $f^{-1}(Z)=\{x\in\mathbb{R}^n : f(x)=0\}$ of $\mathbb{R}^n$ (if it is non-empty). By Remark 11.10 (a) it suffices for this that $\operatorname{rk}_a f=k$ at all points $a\in\mathbb{R}^n$ with $f(a)=0$, leading to the standard definition of a submanifold of $\mathbb{R}^n$ [G1, Definition 27.18].

For example, the sphere $S^n=\{x\in\mathbb{R}^{n+1} : \|x\|=1\}$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$ as it is the inverse image of the origin under the morphism

$$f\colon\mathbb{R}^{n+1}\to\mathbb{R},\ x\mapsto\|x\|^2-1=x_1^2+\cdots+x_{n+1}^2-1,$$

which clearly has rank 1 at all points of the sphere.

(b) As a non-trivial example of Proposition 11.9 (b) we can take the morphism

$$f\colon\mathbb{R}^n\to\mathbb{R}^{n+1}=\mathbb{R}^n\times\mathbb{R},\ x\mapsto\left(\frac{2x}{\|x\|^2+1},\frac{\|x\|^2-1}{\|x\|^2+1}\right),$$

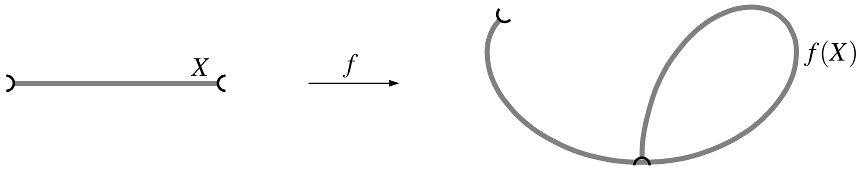
which we had seen already in Example 10.4 (b) as the inverse of the chart

$$\varphi\colon S^n\backslash\{N\}\to\mathbb{R}^n,\ (x,y)\mapsto\frac{x}{1-y}$$

on $S^n\subset\mathbb{R}^{n+1}=\mathbb{R}^n\times\mathbb{R}$ obtained by projection from the north pole $N=(0,1)$. As $\varphi\circ f$ is the identity on $\mathbb{R}^n$, the rank of f must be n everywhere, i. e. f is an immersion. Moreover, we know that f is a homeomorphism onto its image $S^n\backslash\{N\}$ since φ is a chart for S^n. Hence, Proposition 11.9 (b) applies and shows that $\mathbb{R}^n$ is actually isomorphic to $S^n\backslash\{N\}$, i. e. (together with the corresponding statement for the other chart leaving out the south pole) that the manifold structure defined in Example 10.4 (b) is the same as the one obtained by Construction 11.2 for the submanifold S^n of $\mathbb{R}^{n+1}$.

(c) The picture below shows the morphism

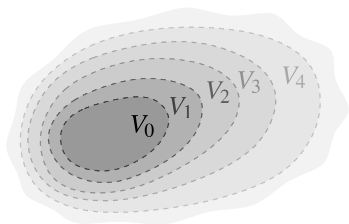
$$f\colon X=\left(0,\frac{3}{4}\pi\right)\to\mathbb{R}^2,\ t\mapsto(\sin 2t\sin t,\sin 2t\cos t).$$



Without even looking at the formula, one can see from the picture already why the condition of f being a homeomorphism between Z and $f(Z)$ in Proposition 11.9 (b) can in general not be replaced by the weaker assumption of injectivity: The map f is an immersion and injective, but its image $f(X)$ is not a 1-dimensional submanifold of $\mathbb{R}^2$ as there is no local chart to $\mathbb{R}$ around the origin for the same reason as in Example 10.4 (e).

Having introduced submanifolds, let us now pursue Example 11.11 (a) a little further: Probably, when visualizing manifolds you would want to think of them as submanifolds of $\mathbb{R}^n$ for some $n \in \mathbb{N}$, even if they have not been defined that way originally – such as e. g. for the Möbius strip or the torus in Example 1.17 where we have drawn pictures in 3-space to show their topology. In addition to visualization, this also has the advantage that it gives us a way to assign global and not just local coordinates to the points on a manifold. So let us ask the question: Can any manifold be embedded in some $\mathbb{R}^n$, i. e. is it always isomorphic to a submanifold of $\mathbb{R}^n$? It turns out that the answer to this question is yes, but given that the definition of a manifold contains almost no global conditions at all it should not come as a surprise that this statement is quite deep and requires some technical preparation.

In fact, the only global requirements for manifolds are the second countability and the Hausdorff axiom, so let us start by studying their implications. We want to show first that these conditions ensure that a manifold X is not too big in the sense that it has an "onion structure" as in the picture on the right: There is a sequence $V_0 \subset \overline{V_0} \subset V_1 \subset \overline{V_1} \subset V_2 \subset \cdots$ covering X with each V_k open and $\overline{V_k}$ compact, so in particular X can be exhausted by countably many compact subspaces.



Lemma 11.12. *Let X be a manifold.*

(a) *Every point of X has a compact neighborhood.*

(b) *There is a countable basis $(B_i)_{i\in I}$ for the topology of X such that every closure $\overline{B_i} \subset X$ for $i \in I$ is compact.*

Proof.

(a) Let $a \in X$ be any point. As the statement is local, we may assume that $X \subset \mathbb{R}^n$ is an open subset. Then $U_\varepsilon(a) \subset X$ for some $\varepsilon > 0$, and thus $\overline{U_{\varepsilon/2}(a)}$ is a compact neighborhood of a.

(b) As X is second countable, there is a countable basis $(B_j)_{j\in J}$ for the topology of X. We set $I := \{i \in J : \overline{B_i} \text{ is compact}\}$ and claim that $(B_i)_{i\in I}$ is a basis as well.

To prove this, we show that any B_j for $j \in J\backslash I$ can already be covered by suitable basis elements B_i for $i \in I$ and thus can be dropped from the basis. Consider an arbitrary point $a \in B_j$, and pick a compact neighborhood K of it by (a). Then a lies in the open set $B_j \cap \mathring{K}$, hence there is a basis element B_{i_a} for $i_a \in J$ with $a \in B_{i_a} \subset B_j \cap \mathring{K}$. In fact, we even have $i_a \in I$: Taking closures, we see that $\overline{B_{i_a}} \subset \overline{K} = K$ since K is closed by Remark 10.11 (b), hence $\overline{B_{i_a}}$ is compact as a closed subset of a compact space [G3, Proposition 4.10 (b)]. Taking these open sets for all $a \in B_j$ together, we obtain $B_j = \bigcup_{a\in B_j} B_{i_a}$ as claimed. □

Proposition 11.13 (Onion structure of manifolds)**.** *For any manifold X, there is a sequence $(V_k)_{k\in\mathbb{N}}$ of open subsets of X with*
$$V_0 \subset \overline{V_0} \subset V_1 \subset \overline{V_1} \subset V_2 \subset \cdots$$
such that each $\overline{V_k}$ is compact and $\bigcup_{k\in\mathbb{N}} V_k = X$.

Proof. By Lemma 11.12 (b) we can pick a countable basis $(B_i)_{i\in\mathbb{N}}$ for the topology of X such that every closure $\overline{B_i}$ is compact. We now construct the sequence $(V_k)_{k\in\mathbb{N}}$ recursively, starting with $V_0 := \emptyset$.

For the induction step, assume that V_k has already been constructed for some $k \in \mathbb{N}$. As $\overline{V_k}$ is compact, there is a finite set $I_k \subset \mathbb{N}$ with $\overline{V_k} \subset \bigcup_{i\in I_k} B_i$, for which we may assume that $\{0,\dots,k\} \subset I_k$. Set $V_{k+1} := \bigcup_{i\in I_k} B_i$. Then $\overline{V_k} \subset V_{k+1}$ by construction, V_{k+1} is open and $\overline{V_{k+1}} = \bigcup_{i\in I_k} \overline{B_i}$ as a finite union of compact spaces compact, and the condition $\{0,\dots,k\} \subset I_k$ ensures that $\bigcup_{k\in\mathbb{N}} V_k = \bigcup_{i\in\mathbb{N}} B_i = X$. □

Example 11.14.

(a) Of course, for a compact manifold X the statement of Proposition 11.13 is trivial since we can just take $V_k = X$ for all $k \in \mathbb{N}$.

(b) For $X = \mathbb{R}^n$, a possible choice of onion structure would be to take open balls $V_k = U_k(0)$ for all $k \in \mathbb{N}$. For $X = \mathbb{R}^n\backslash\{0\}$, we could take $V_0 = \emptyset$ and $V_k = \{x \in \mathbb{R}^n : \frac{1}{k} < \|x\| < k\}$ for $k > 0$.

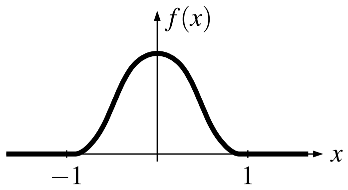
Onion structures serve several purposes; they will e. g. allow us to establish a manifold version of Mayer-Vietoris induction once we have studied de Rham cohomology (see Lemma 15.6). For the moment, we just need some part of the Mayer-Vietoris picture of Chapter 7, namely a dual analytic version of subdivisions. Recall that to study the singular homology of a union $U \cup V$ of open subsets in a topological space we had to subdivide singular simplices in $U \cap V$ fine enough so that each part of the subdivision lies in U or V. Let us now establish an analogous statement for functions: Roughly speaking, given a smooth function $f\colon U\cup V \to \mathbb{R}$ on a union of two open subsets of a manifold, we want to write it as $f = f_U + f_V$, where f_U and f_V are zero away from U and V, respectively. Of course, both f_U and f_V should still be smooth on $U \cup V$, so a naive solution along the lines of

$$f_U(x) := \begin{cases} f(x) & \text{if } x \in U, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f_V(x) := \begin{cases} f(x) & \text{if } x \in V\backslash U, \\ 0 & \text{otherwise} \end{cases}$$

is not what we are looking for. Instead, we need to construct a function f_U that interpolates smoothly between f on $U\backslash V$ and 0 on $V\backslash U$, and similarly for f_V, so that their sum still equals f. Note that it is sufficient to do this for the constant function 1, as for the general case we can then just multiply our interpolating functions with f. Such a decomposition of the constant function 1 into parts that are non-zero only at specific places is called a *partition of unity*, a crucial concept in manifold theory that we will introduce now – in fact, not just for unions of two subsets, but for arbitrary open covers. Its construction is based on the following so-called *bump functions*.

Construction 11.15 (Bump functions and functions with compact support)**.** Recall from analysis that there are non-zero smooth functions $f\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ that are identically zero outside a given range [G1, Exercise 11.12], such as e. g.

$$f\colon \mathbb{R} \to \mathbb{R},\; x \mapsto \begin{cases} \exp\left(\dfrac{1}{x^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1 \end{cases}$$



as shown in the picture on the right. In other words, this function approaches the horizontal axes infinitely flat to any order at the points $x = \pm 1$. It is usually given as an example for a *non-analytic* function, i. e. a function that cannot be represented by its Taylor series. 18

It is easy to extend this construction to manifolds. First of all, for a real-valued function $f\colon X \to \mathbb{R}$ on a manifold X we call

$$\operatorname{supp} f := \overline{f^{-1}(\mathbb{R}\backslash\{0\})} \quad \subset X$$

the **support** of f. Note that we are taking *the closure of* the set of all points with non-zero values here, so e. g. the function above has support $\operatorname{supp} f = [-1,1]$. The special feature of this function is that its support is actually compact – most "interesting" functions $f\colon \mathbb{R} \to \mathbb{R}$ that you would think of will probably have $\operatorname{supp} f = \mathbb{R}$ and thus non-compact support.

However, modeled on the function above we can easily construct smooth functions with compact support on arbitrary manifolds: Let X be an n-dimensional manifold, and let $a \in X$ be an arbitrary point with a given open neighborhood $U \subset X$. By a **bump function** for a in U we will mean a smooth function $f\colon X \to \mathbb{R}_{\geq 0}$ with $f(a) > 0$ and compact support $\operatorname{supp} f \subset U$. To obtain such a bump function, we can shrink U to a smaller open neighborhood of a such that there is a chart $\varphi\colon U \to V$ with V an open ball in $\mathbb{R}^n$; by a coordinate transformation on $\mathbb{R}^n$ we can furthermore assume that $V = U_2(0)$ is the ball around the origin with radius 2. Extending the function in the picture above to the multidimensional case and pulling it back by the chart φ then gives us a bump function

$$f\colon X \to \mathbb{R},\ x \mapsto \begin{cases} \exp\left(\dfrac{1}{\|\varphi(x)\|^2 - 1}\right) & \text{if } x \in U \text{ with } \|\varphi(x)\| < 1, \\ 0 & \text{otherwise} \end{cases}$$

as desired: It is smooth (first on U in the same way as the bump function in the picture above, and then also by extension by zero on all of X), and its support is $\varphi^{-1}(K_1(0))$, which is compact and contained in $\varphi^{-1}(U_2(0)) = U$.

Note again that taking the closure is important in this construction: The function f in the picture above is a bump function for 0 in $(-2,2)$, but not for 0 in $(-1,1)$ since its support $[-1,1]$ is not contained in $(-1,1)$. Intuitively, for a bump function for $a \in U$ it is not sufficient that f is zero outside U, it must in fact already be identically zero in a neighborhood of the boundary of U.

Proposition 11.16 (Partitions of unity)**.** *Let $(U_i)_{i\in I}$ be an open cover of a manifold X. Then there exists a (not necessarily finite) family $(f_j)_{j\in J}$ of smooth functions $f_j\colon X \to \mathbb{R}_{\geq 0}$ such that:*

(a) *The support of every function f_j for $j \in J$ is compact and contained in one of the open subsets U_i.*

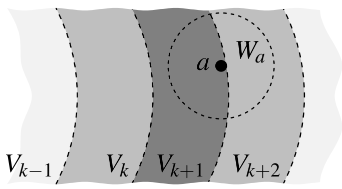
(b) *Every point $a \in X$ has an open neighborhood on which almost all f_j are identically zero – this ensures that the sum $\sum_{j\in J} f_j\colon X \to \mathbb{R}$ is locally finite, i. e. there are no convergence issues when studying local properties of this sum.*

(c) $\sum_{j\in J} f_j = 1$.

Such a family $(f_j)_{j\in J}$ is called a ***partition of unity*** *subordinate to the open cover $(U_i)_{i\in I}$.*

Proof. First of all note that we can replace (c) by the weaker condition that the sum $g := \sum_{j\in J} f_j$ is nowhere zero, i. e. that at least one f_j is non-zero at any point of X: We can then just normalize all functions by replacing each f_j by $\frac{f_j}{g}$.

To construct the family $(f_j)_{j\in J}$, pick an onion structure $(V_k)_{k\in\mathbb{N}}$ as in Proposition 11.13, setting $V_k = \emptyset$ for $k < 0$. For a fixed $k \in \mathbb{N}$ and any a in the compact set $\overline{V_{k+1}}\backslash V_k$ (shown in dark gray in the picture on the right), pick $i \in I$ with $a \in U_i$, and choose a bump function f_a for a in the open set $U_i \cap (V_{k+2}\backslash\overline{V_{k-1}})$ as in Construction 11.15. We denote by $W_a = f_a^{-1}(\mathbb{R}\backslash\{0\})$ the open set on which it is non-zero, contained in the dark and light gray region in the picture.



Note that these open sets W_a cover the compact set $\overline{V_{k+1}}\backslash V_k$, so we can pick a finite set $J_k \subset \overline{V_{k+1}}\backslash V_k$ such that $\overline{V_{k+1}}\backslash V_k \subset \bigcup_{a\in J_k} W_a$. Finally, set $J = \bigcup_{k\in\mathbb{N}} J_k$.

Then $(f_a)_{a\in J}$ satisfies the condition of the proposition: Property (a) holds for every f_a by construction. The sum $\sum_{a\in J} f_a$ is locally finite as required by (b) since every $x \in X$ is contained in some open set $V_{k+2}\backslash\overline{V_k}$, and only the functions f_a with a in the finite set $J_{k-1} \cup J_k \cup J_{k+1} \cup J_{k+2}$ can have non-zero values there. Finally, every $x \in X$ is contained in some compact set $\overline{V_{k+1}}\backslash V_k$, and by definition there will then be some $a \in J_k$ with $f_a(x) > 0$, showing (c) as remarked at the beginning of the proof. $\square$

In fact, if one does not care about the support of the individual functions being compact, there is also a simpler version of partitions of unity that needs only one function for each open set in the given cover.

Corollary 11.17 (Alternative version of partitions of unity)**.** *Again let $(U_i)_{i\in I}$ be an open cover of a manifold X. Then there exists a family $(f_i)_{i\in I}$ of smooth functions $f_i\colon X\to\mathbb{R}_{\geq 0}$ labeled by the same index set I such that:*

(a) *The support of every function f_i for $i\in I$ is contained in U_i (however not necessarily compact).*

(b) *Every point $a\in X$ has an open neighborhood on which almost all f_i are identically zero.*

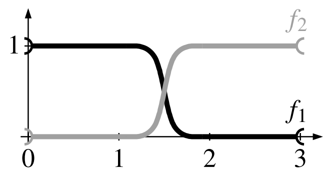
(c) $\sum_{i\in I} f_i = 1$.

Again, such a family $(f_i)_{i\in I}$ will be called a partition of unity subordinate to *$(U_i)_{i\in I}$.*

Proof. Take a partition of unity $(g_j)_{j\in J}$ as in Proposition 11.16, pick for each $j\in J$ an index $i_j\in I$ with $\operatorname{supp} g_j\subset U_{i_j}$, and set $f_i := \sum_{j\in J: i_j=i} g_j$ for all $i\in I$. (Note that there might and generally will be infinitely many j contributing to this sum, but not locally by Proposition 11.16 (b) – hence the local conditions (b) and (c) of the proposition carry over to the new family $(f_i)_{i\in I}$, whereas compactness of the support does not.) $\square$

Example 11.18. Let $X=(0,3)\subset\mathbb{R}$, and consider the open cover of X by the two open subsets $U_1=(0,2)$ and $U_2=(1,3)$.

(a) A partition of unity for X subordinate to the given cover in the sense of Corollary 11.17, i. e. consisting of two non-negative smooth functions f_1 and f_2 with $f_1+f_2=1$, and having their support in U_1 and U_2, respectively, is easy to find without going through the construction in the proof; a graphical visualization is shown in the picture on the right.



(b) Note however that the functions of (a) do not have compact support due to the interval X being open. If we insist on a partition of unity consisting of functions with compact support as in Proposition 11.16, we have to replace e. g. f_1 by countably many functions – for example, by functions $f_{1,n}$ with $n\in\mathbb{N}_{>0}$ having support in the overlapping intervals $\left[\frac{c}{n+2},\frac{c}{n}\right]$ for a suitable $c\in(1,2)$, normalized so that their (locally finite) sum is 1.

Remark 11.19. Of course, dealing with infinite partitions of unity as in Example 11.18 (b) is much more uncomfortable than with finite ones as in (a). Note however that for compact manifolds we can achieve the best of both worlds: Even in Corollary 11.17 the supports of the functions f_j will then be compact since they are closed subsets of the compact space X [G3, Proposition 4.10 (b)].

Exercise 11.20. Let $U_i := \{(x_1,x_2)\in\mathbb{R}^2 : x_i>0\}\subset\mathbb{R}^2$ for $i\in\{1,2\}$.

Find an explicit partition of unity (f_1,f_2) for the submanifold $X := U_1\cup U_2$ of $\mathbb{R}^2$ subordinate to the open cover (U_1,U_2) in the sense of Corollary 11.17.

We are now ready to use partitions of unity to prove our final result of this chapter:

Proposition 11.21 (**Whitney embedding**)**.** *Every compact manifold is isomorphic to a submanifold of $\mathbb{R}^N$ for some $N\in\mathbb{N}$.*

Proof. Let X be a compact n-dimensional manifold. Due to the compactness, we can cover X by finitely man charts $\varphi_i\colon U_i\to V_i$ for $i=1,\dots,k$. Moreover, we pick a partition of unity subordinate to this cover in the sense of Corollary 11.17, given by the smooth functions $f_i\colon X\to V_i\subset\mathbb{R}^n$ with support in U_i.

Now consider the map $F\colon X\to\mathbb{R}^{k(n+1)}=\mathbb{R}^{n+1}\times\cdots\times\mathbb{R}^{n+1}$ with the component functions

$$F_i\colon X\to\mathbb{R}^{n+1}=\mathbb{R}^n\times\mathbb{R},\ x\mapsto (f_i(x)\,\varphi_i(x),f_i(x)) := \begin{cases}(f_i(x)\,\varphi_i(x),f_i(x)) & \text{if } x\in U_i,\\ 0 & \text{otherwise}\end{cases}$$

for $i=1,\dots,k$. Note that despite their piecewise definition all F_i are smooth since they are identically zero in the open neighborhood $X\backslash\operatorname{supp} f_i$ of ∂U_i.

We claim that F is an isomorphism onto its image $F(X)$ as a submanifold of $\mathbb{R}^N := \mathbb{R}^{k(n+1)}$. By Proposition 11.9 (b) and Remark 11.10 (b), it suffices to prove that F is an injective immersion:

(a) F is injective: Let $x,y \in X$ with $F(x) = F(y)$; in particular, we then have $f_i(x) = f_i(y)$ for all $i = 1,\dots,k$. As $(f_i)_i$ is a partition of unity, this number must be non-zero for at least one i (which also requires $x,y \in U_i$). But for this i we then have $f_i(x)\,\varphi_i(x) = f_i(y)\,\varphi_i(y)$ as well, i. e. $\varphi_i(x) = \varphi_i(y)$. As $\varphi_i \colon U_i \to V_i$ is an isomorphism, this means that $x = y$.

(b) F is an immersion: Let $x \in X$, and again pick $i \in \{1,\dots,k\}$ with $f_i(x) > 0$, implying $x \in U_i$. Then with respect to the chart φ_i we have

$$F_i \circ \varphi_i^{-1} \colon V_i \to \mathbb{R}^{n+1},\ y \mapsto (\tilde{f}_i(y)\,y, \tilde{f}_i(y)) \quad \text{with } \tilde{f} := f \circ \varphi_i^{-1},$$

which has Jacobian matrix

$$(F_i \circ \varphi^{-1})'(y) = \begin{pmatrix} \tilde{f}_i'(y)\,y + \tilde{f}_i(y)\,E \\ \tilde{f}_i'(y) \end{pmatrix} \quad \in \mathbb{R}^{(n+1)\times n}$$

at $y = \varphi_i(x)$. By row transformations we can bring this into the form

$$\begin{pmatrix} \tilde{f}_i(y)\,E \\ \tilde{f}_i'(y) \end{pmatrix} \quad \in \mathbb{R}^{(n+1)\times n},$$

which has rank n since $\tilde{f}_i(y) = f_i(x) > 0$. Hence, F_i and thus also F has rank n at x, i. e. F is an immersion as required. □

Remark 11.22.

(a) The Whitney embedding ensures that compact manifolds such as projective spaces that do not naturally come as submanifolds of some $\mathbb{R}^N$ can nevertheless be written in that way – which is helpful for our further study of compact manifolds since being able to assume an embedding in $\mathbb{R}^N$ will often simplify arguments (see e. g. Construction 12.13). However, the explicit construction of this embedding in the proof of Proposition 11.21 is rarely useful since it uses partitions of unity and a rather high-dimensional ambient space.

(b) Actually, the Whitney embedding statement is more general than what we have shown in Proposition 11.21. In addition, one can prove [B, Theorem II.10.8]:

- The statement also holds without the compactness assumption, i. e. any manifold can be embedded as a submanifold of $\mathbb{R}^N$ (in fact, even as a closed submanifold).
- There is a bound on the necessary ambient dimension in the sense that every manifold can be embedded in $\mathbb{R}^N$ with $N \le 2\dim X$ (see Exercise 1.35 for a simplicial version).

For us, Proposition 11.21 will suffice however, so we will not prove these generalizations.

(c) Even if it is not the topic of this class, let us quickly mention that some very interesting phenomena occur when considering the analogous situation over the complex numbers. Recall from Remark 10.3 (c) that complex manifolds could be defined in exactly the same way as real manifolds, replacing real by complex differentiability. In many aspects, they also behave in the same way: One could literally copy the definition of morphisms, ranks of maps, and submanifolds, and also Proposition 11.9 regarding inverse images and images of submanifolds under submersions and immersions, respectively, holds without change (basically because the inverse function theorem, which was the main ingredient for this statement, holds over $\mathbb{C}$ as well). However, as you might know from complex analysis, bump functions as in Construction 11.15 simply do not exist in the complex world since every holomorphic function is analytic [G4, Proposition 7.10]. For the same reason, partitions of unity are impossible over $\mathbb{C}$, and in fact, the Whitney embedding statement of 11.21 is false for complex manifolds. More drastically, a compact complex manifold (of positive dimension) can *never* be embedded into $\mathbb{C}^N$! A compelling reason for this if you know some basic complex analysis is the Maximum Modulus Principle [G4, Proposition 6.14], which also has a multivariate version: The coordinate functions of an embedding $f\colon X \to \mathbb{C}^N$ of a compact complex manifold X into $\mathbb{C}^N$ would clearly be bounded holomorphic functions due to the compactness of X, hence would have to have a local maximum somewhere – which is impossible unless f is constant, i. e. not an embedding.

Exercise 11.23.

(a) Let $(f_n)_{n\in\mathbb{N}}$ be a family of smooth functions $f_n\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ with compact support. Show that there are constants $c_n \in \mathbb{R}_{>0}$ for $n \in \mathbb{N}$ such that the sum $f := \sum_{n=0}^{\infty} c_n f_n\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ converges and is a smooth function as well.

(Hint: It is known from standard analysis that a sequence $g_n\colon \mathbb{R} \to \mathbb{R}$ of differentiable functions converges to a differentiable function with derivative $\lim\limits_{n\to\infty} g_n'$ if both $(g_n)_{n\in\mathbb{N}}$ and $(g_n')_{n\in\mathbb{N}}$ converge uniformly.)

(b) Show that every closed subset $A \subset \mathbb{R}$ is the zero locus of a smooth function, i. e. that there is a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ with $A = \{x \in \mathbb{R} : f(x) = 0\}$.

(Hint: Use second countability and suitable bump functions for the complement $\mathbb{R}\backslash A$.)