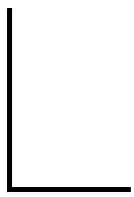
# Smooth Manifolds

## 10. Manifolds

Up to now in these notes we have studied the geometry of topological spaces using mainly algebraic methods. As already announced in the introduction, let us now change our point of view entirely and switch to analysis. We have already seen in Example 5.4 and Remark 5.5 that the simplicial coboundary map can be interpreted as a discrete version of differentiation, and we now want to set up an analytic theory of cohomology in which the coboundary map actually takes derivatives of functions in the usual sense. The so-called *de Rham cohomology* obtained in this way will again turn out to be compatible with our previous constructions, i. e. it will be naturally isomorphic to singular cohomology whenever both are defined.

But before we can construct this analytic cohomology in Chapter 13, we have to face a fundamental problem: Which geometric spaces do we want to consider? So far you probably only know how to differentiate functions on open subsets of $\mathbb{R}^n$, but this class of spaces looks very restrictive for our purposes. On the other hand, arbitrary topological spaces (as in the singular theory) clearly do not carry enough structure to allow a definition of derivatives. The key observation to solve this problem is that differentiation is a local construction, and thus it should be fine to consider topological spaces that locally look like $\mathbb{R}^n$. Such spaces will be called *manifolds*, and we will introduce and study them in the next three chapters – completely independently of (co-)homology, for the time being.

So what exactly does it mean to "locally look like $\mathbb{R}^n$"? Does the L-shaped subspace of $\mathbb{R}^2$ shown in the picture on the right look like an interval, i. e. locally like $\mathbb{R}$? It is clearly *homeomorphic* to an interval, so from a topological point of view the answer would be yes. However, from a differentiable point of view, the corner point seems to make a difference. Hence it will not be sufficient for our purposes to define a manifold to be just a topological space locally homeomorphic to $\mathbb{R}^n$, i. e. such that every point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.



However, the idea of homeomorphisms $\varphi\colon U \to V$ from open subsets $U$ of our topological space $X$ that we want to consider to open subsets $V$ of $\mathbb{R}^n$ is still useful: Choosing such a homeomorphism $\varphi$ allows us to associate coordinates in $\mathbb{R}^n$ to points of $U$, and thus to define e. g. the differentiability of functions on $U$ with respect to these coordinates. We will call $\varphi$ a *(coordinate) chart* for that reason. But these charts will be additional given data, and of course we will need several of them to cover all points of $X$. Hence, we have to make sure that this definition of differentiability does not depend on the chosen chart, i. e. that our given charts are compatible in a certain sense. Let us now say exactly
15 what we mean by this.

**Notation 10.1** (Smooth maps and diffeomorphisms)**.** Let $\varphi\colon U \to V$ be a map between open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ for $m,n \in \mathbb{N}$.

(a) We say that $\varphi$ is **smooth** if it is infinitely differentiable. In the case $V = \mathbb{R}$, the vector space of all smooth functions from $U$ to $\mathbb{R}$ will be written as $\Omega^0(U)$. (This corresponds to such functions being "differential forms of degree 0", a concept to be introduced in Construction 13.1 (a)). Another standard notation in analysis for $\Omega^0(U)$ is clearly $C^\infty(U)$, but we will avoid this notation here since the letter $C$ is ubiquitous in (co-)homology theory already.

In particular, for an arbitrary open subset $V \subset \mathbb{R}^m$ the map $\varphi\colon U \to V$ is smooth if and only if all its component functions are smooth, i. e. lie in $\Omega^0(U)$.

(b) The map $\varphi$ is called a **diffeomorphism** if it is bijective, and both $\varphi$ and $\varphi^{-1}$ are smooth. (Note that sometimes in the literature a diffeomorphism only requires $\varphi$ and $\varphi^{-1}$ to be continuously differentiable once. Also note that a diffeomorphism clearly requires $m = n$ as the Jacobian matrices of $\varphi$ and $\varphi^{-1}$ need to be inverses of each other.)

**Definition 10.2** (Charts and atlases)**.** Let $X$ be a topological space, and let $n \in \mathbb{N}$.

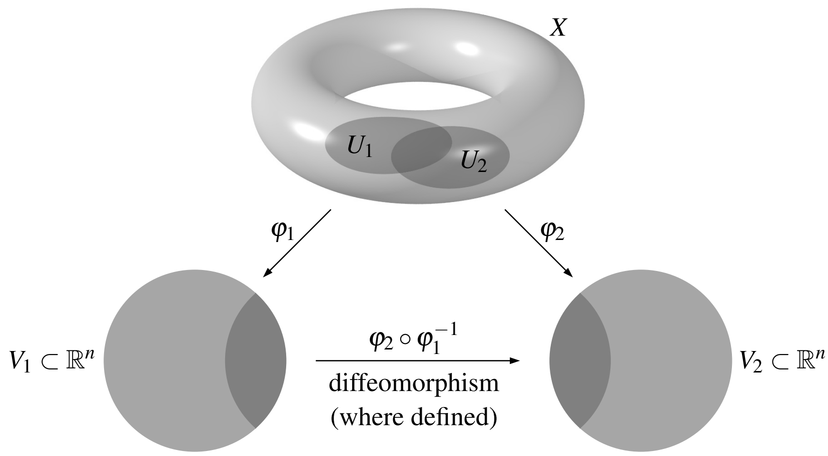
(a) As in the picture below, an ($n$-dimensional coordinate) **chart** on $X$ is a homeomorphism $\varphi\colon U \to V$ from an open subset $U \subset X$ to an open subset $V \subset \mathbb{R}^n$. For a point $x \in U$, we call $\varphi(x) \in \mathbb{R}^n$ the **(local) coordinates** of $x$ in this chart.

(b) We say that two $n$-dimensional charts $\varphi_1\colon U_1 \to V_1$ and $\varphi_2\colon U_2 \to V_2$ are **compatible** if the **transition map**

$$\varphi_2 \circ \varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

(such as between the dark circle segments in the picture) is a diffeomorphism.

(c) An ($n$-dimensional) **atlas** on $X$ is a family $(\varphi_i)_{i\in I}$ of pairwise compatible $n$-dimensional charts $\varphi_i\colon U_i \to V_i$ covering $X$, i. e. with $\bigcup_{i\in I} U_i = X$.




**Remark 10.3.**

(a) Note the subtle but crucial difference between homeomorphisms and diffeomorphisms in Definition 10.2: A priori, $X$ is only a topological space, so it would not make sense to require the charts to be diffeomorphisms. Instead, as mentioned already in the introduction we will use these charts to define a notion of differentiability on $X$. In order for this notion to be well-defined, we will need that the transition maps (between open subsets of $\mathbb{R}^n$) are diffeomorphisms – homeomorphisms would not suffice here. In fact, the maps $\varphi_2 \circ \varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ in Definition 10.2 (b) are already homeomorphisms by construction, so their smoothness is the only condition to be checked here.

(b) As you can see, our setup in these notes will be geared towards *infinite differentiability* only. This will allow us to define derivatives of any order on $X$, and has the additional advantage that differentiation does not degrade the order of differentiability. However, one could equally well fix a natural number $k \in \mathbb{N}$ and require the transition functions in Remark 10.2 (b) to only be $k$-fold differentiable, or $k$-fold continuously differentiable, leading to different notions of compatibility, atlases, and finally manifolds, that might be useful for other purposes. It would even be possible to choose $k = 0$, in which case condition (b) in Definition 10.2 is trivial as already remarked in (a), and we would obtain so-called *topological manifolds*.

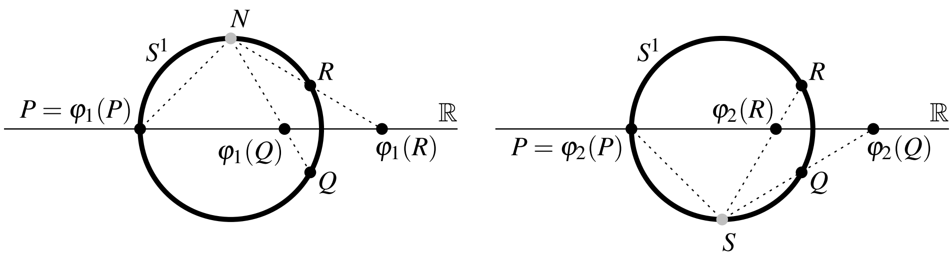
(c) If you know some complex analysis, another variant of Definition 10.2 would be to replace $\mathbb{R}$ by $\mathbb{C}$, and thus consider complex instead of real differentiability (i. e. transition functions that are holomorphic maps). The *complex manifolds* obtained in that way are clearly very important objects in mathematics, and share the same interesting flavor with one-dimensional complex analysis that their initial definitions are identical to the real case, yet their behavior very different. In these notes we will see a few complex examples such as e. g. the complex projective spaces of Construction 8.3, but in order not to assume any complex analysis we will always consider them over $\mathbb{R}$, so that they will become real $2n$-dimensional manifolds instead of complex $n$-dimensional ones.

**Example 10.4.**

(a) Every open subset $X \subset \mathbb{R}^n$ admits a natural $n$-dimensional atlas with the identity map $\mathrm{id}_X$ as the only chart.

(b) Consider the $n$-dimensional sphere $S^n = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : ||x||^2 + y^2 = 1\} \subset \mathbb{R}^{n+1}$. Removing the north pole $N := (0,1)$ or the south pole $S := (0,-1)$, we clearly obtain two open subsets homeomorphic to $\mathbb{R}^n$ that cover $S^n$. In fact, homeomorphisms $\varphi_1 : S^n\backslash\{N\} \to \mathbb{R}^n$ and $\varphi_2 : S^n\backslash\{S\} \to \mathbb{R}^n$ can be obtained by projecting from $N$ or $S$ onto $\mathbb{R}^n \times \{0\}$, respectively, as shown in the picture below for $n = 1$. The explicit formulas for these maps are

$$\varphi_1 : S^n\backslash\{N\} \to \mathbb{R}^n,\ (x,y) \mapsto \frac{x}{1-y} \quad \text{and} \quad \varphi_2 : S^n\backslash\{S\} \to \mathbb{R}^n,\ (x,y) \mapsto \frac{x}{1+y}.$$




From this it is a straightforward calculation to obtain the inverse of e. g. $\varphi_1$ as

$$\varphi_1^{-1} : \mathbb{R}^n \to S^n\backslash\{N\},\ x \mapsto \left(\frac{2x}{||x||^2+1}, \frac{||x||^2-1}{||x||^2+1}\right),$$

and thus the transition map between the two charts as

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^n\backslash\{0\},\ x \mapsto \frac{x}{||x||^2}.$$

This map is clearly smooth, so $\varphi_1$ and $\varphi_2$ form an $n$-dimensional atlas on $S^n$.

(c) We have already seen in Construction 8.3 that the (real or complex) projective spaces $\mathbb{P}^n_{\mathbb{K}}$ can be covered by the open subsets $U_i = \{(x_0 : \cdots : x_n) : x_i \neq 0\}$ for $i = 0,\dots,n$, which admit charts

$$\varphi_i : U_i \to \mathbb{K}^n,\ (x_0 : \cdots : x_n) \mapsto \left(\frac{x_0}{x_i},\dots,\widehat{\frac{x_i}{x_i}},\dots,\frac{x_n}{x_i}\right).$$

A simple computation shows again that the transition maps between these charts are smooth, so $\varphi_0,\dots,\varphi_n$ form an atlas on $\mathbb{P}^n_{\mathbb{K}}$ (of real dimension $n$ or $2n$ in the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, respectively).

(d) The (non-differentiable) function

$$\varphi : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} 2x & \text{if } x \geq 0, \\ x & \text{if } x \leq 0 \end{cases}$$

is a homeomorphism, and thus it is a chart defining a 1-dimensional atlas on $\mathbb{R}$ – note that the condition of Definition 10.2 (b) is empty for atlases containing only one chart. However, this chart is incompatible with the natural identity chart of (a), so we will usually not consider it.

(e) Both the interval $X = [0,1] \subset \mathbb{R}$ and the union $Y = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ or } x_2 = 0\}$ of the two coordinate axes in $\mathbb{R}^2$ do not admit a 1-dimensional atlas: If we remove the zero point from any connected open neighborhood of it, it will remain connected for $X$ and separate into four connected components for $Y$, whereas removing a point from a connected open subset of $\mathbb{R}$ yields exactly two connected components. Hence, there can be no homeomorphism between these spaces, i. e. no chart containing the zero point.

However, $X$ will be a 1-dimensional *manifold with boundary*, a concept that we will introduce in Construction 14.3.

Let us now see how we can use atlases to define concepts of differentiability on topological spaces. The setting here is similar to matrix computations in linear algebra: When considering linear maps between vector spaces, computations are easily done in terms of matrices for maps $K^n \to K^m$ with $m,n \in \mathbb{N}$. To transfer these methods to a morphism $f : V \to W$ between arbitrary vector spaces with $\dim V = n$ and $\dim W = m$, one uses coordinate maps $\varphi : V \to K^n$ and $\psi : W \to K^m$ and studies the linear map $\psi \circ f \circ \varphi^{-1} : K^n \to K^m$ instead that sends the coordinate vector of $x \in V$ to the coordinate vector of $f(x) \in W$. As this is now a morphism between $K^n \to K^m$, one can use matrix techniques again for computations. We will apply the same idea now to carry over notions of differentiability from $\mathbb{R}^n$ to topological spaces with atlases, using charts as coordinate maps:

**Construction 10.5** (Smooth maps between topological spaces with atlases)**.** Let $X$ and $Y$ be topological spaces with atlases $(\varphi_i)_{i\in I}$ and $(\psi_j)_{j\in J}$ of dimensions $n$ and $m$, respectively. Moreover, let $f : X \to Y$ be an arbitrary map.

(a) For a given point $a \in X$ let $\varphi_i : U_i \to V_i$ with $i \in I$ be a chart with $a \in U_i$, and let $\psi_j : U'_j \to V'_j$ with $j \in J$ be a chart with $f(a) \in U'_j$. We can then form the composition

$$f_{i,j} := \psi_j \circ f \circ \varphi_i^{-1} : V_i \to V'_j$$

that locally around $a$ sends the coordinates of a point $x \in U_i$ to the coordinates of $f(x) \in U'_j$ in these charts and is a map between open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. In this situation, we call $f$ **smooth** at the point $a$ if $f_{i,j}$ is smooth at its coordinate point $\varphi_i(a)$. Note that this is well-defined by the chart compatibility condition of Definition 10.2: If we pick different charts $\varphi_{\tilde{i}}$ and $\psi_{\tilde{j}}$ containing $a$ and $f(a)$, respectively, we have

$$f_{\tilde{i},\tilde{j}} = \psi_{\tilde{j}} \circ f \circ \varphi_{\tilde{i}}^{-1} = \underbrace{\psi_{\tilde{j}} \circ \psi_j^{-1}}_{(2)} \circ \underbrace{\psi_j \circ f \circ \varphi_i^{-1}}_{=f_{i,j}} \circ \underbrace{\varphi_i \circ \varphi_{\tilde{i}}^{-1}}_{(2)} \tag{1}$$

where defined. As the transition maps (2) are smooth, the smoothness of $f_{\tilde{i},\tilde{j}}$ is therefore equivalent to the smoothness of $f_{i,j}$ at the point under consideration.

Of course, we say that $f$ is a smooth map if it is smooth at every point $a \in X$: As expected from analysis, smoothness is a pointwise condition.

(b) If $f$ is smooth at $a \in X$, we define the **rank** $\operatorname{rk}_a f$ of $f$ at $a$ to be the rank of the derivative $f'_{i,j}(\varphi_i(a))$, i. e. the rank of the corresponding Jacobian matrix. Again, this is well-defined by (1) since a different choice of charts will just multiply the Jacobian matrix of $f_{i,j}$ with an invertible matrix from the left and right, which does not change its rank.

**Remark 10.6.** In a similar way, we could define what it means that a map between topological spaces with atlases is $k$-fold (continuously) differentiable, but for our purposes we will not need this. Note however that Construction 10.5 only defines what it means that a map is differentiable (resp. smooth), but not what its derivative is in this case! In fact, constructing actual derivatives in a well-defined way is a bit more involved; we will do this in Construction 12.13.

**Example 10.7.** Consider the circle $S^1 \subset \mathbb{R}^2$ with the atlas of Example 10.4 (b). For simplicity, we will restrict our attention to the chart $\varphi_1$ on $S^1 \backslash \{(0,1)\}$ as the computations for $\varphi_2$ are entirely analogous. Recall from the example above that its inverse $\varphi_1^{-1} : \mathbb{R} \to S^1 \backslash \{(0,1)\}$ is given by

$$\varphi_1^{-1}(x) = \left( \frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right) \quad \text{with Jacobian matrix} \quad (\varphi_1^{-1})'(x) = \left( \frac{2-2x^2}{(x^2+1)^2}, \frac{4x}{(x^2+1)^2} \right)^{\mathsf{T}}.$$

(a) The embedding $f: S^1 \to \mathbb{R}^2$ (with the natural atlas on $\mathbb{R}^2$) is smooth, since $f \circ \varphi_1^{-1} = \varphi_1^{-1}$ is clearly a smooth map from $\mathbb{R}$ to $\mathbb{R}^2$. It has rank 1 everywhere as $(\varphi_1^{-1})'(x) \neq 0$ for all $x \in \mathbb{R}$.

(b) The vertical projection $f: S^1 \to \mathbb{R}$, $(x,y) \mapsto x$ is also smooth as $f \circ \varphi_1^{-1}$ is just the second coordinate of $\varphi_1^{-1}$ in this case. Its derivative is $(f \circ \varphi_1^{-1})'(x) = \frac{2-2x^2}{(x^2+1)^2}$, so $f$ has rank 1 everywhere except at the points $(\pm 1, 0)$, where the rank is 0. Geometrically, this means that the linear approximation of the vertical projection at these two points is a constant map, whereas it is non-constant at all other points.

**Exercise 10.8.** Let $f: X \to Y$ be a continuous surjective map between two topological spaces with the following property: For every $a \in Y$ there exists a open neighborhood $U_a$ of $a$ such that $f^{-1}(U_a)$ is a disjoint union of open subsets each homeomorphic to $U_a$ through $f$.

(a) If $Y$ has an atlas, show that there is an atlas on $X$ such that $f$ is smooth.

(b) On the other hand, show by example that if $X$ has an atlas there need not be an atlas on $Y$ such that $f$ is smooth.

From the examples considered so far, it might be tempting to just define a manifold as a topological space $X$ with an atlas. However, there are two slight technical problems with this that we still have to address. The first one is that we really use the atlas only to define what smoothness on $X$ means, and thus the exact form of the charts is just an irrelevant choice of coordinates. For example, we could replace any chart $\varphi: U \to V \subset \mathbb{R}^n$ with $\psi \circ \varphi: U \to V'$ for an arbitrary diffeomorphism $\psi: V \to V' \subset \mathbb{R}^n$ without changing which maps on $X$ are smooth. We could even keep the old chart and add $\psi \circ \varphi$ to the atlas as well, as these two charts are compatible. We could also throw in restrictions of already existing charts, or remove charts if the remaining ones already cover $X$. All these operations clearly change the atlas, but not the information we are actually interested in – and thus we want to say that all these atlases determine the same manifold, which we would not do if a manifold was just a topological space with an atlas. One way to achieve this is the following construction.

**Construction 10.9** (Smooth structures)**.** An atlas $(\varphi_i)_{i \in I}$ on a topological space $X$ is called **maximal** if every chart on $X$ that is compatible with the charts in the atlas is already contained in the atlas, i. e. if the atlas cannot be enlarged without breaking compatibility. This ensures by definition that we cannot modify a maximal atlas in an irrelevant way by adding or removing compatible charts as above. Hence, a maximal atlas exactly captures the idea of encoding which maps on $X$ are smooth, without any arbitrary choices. A maximal atlas on $X$ is therefore also called a **smooth structure** on $X$, and this is what we will require to be given for manifolds.

Yes, a maximal atlas is huge, and we will most definitely never write one down. But every atlas clearly defines a unique maximal one (namely by adding all compatible charts to it), so it will be okay to give an atlas on $X$ as we did e. g. in Example 10.4 to determine a smooth structure on $X$. We just have to remember that another atlas determines the same smooth structure if all charts in both atlases together are compatible. (In fact, an equivalent definition would be that a smooth structure is an equivalence class of atlases, where two atlases are called equivalent if their union is an atlas as well.)

**Remark 10.10** (Charts to open balls or $\mathbb{R}^n$)**.** Assume that we have an $n$-dimensional smooth structure on a topological space $X$. The flexibility in the coordinate charts can actually be used to choose (non-maximal) atlases determining this smooth structure with additional properties, which might sometimes be useful:

(a) We can take a chart $\varphi_a: U_a \to V_a$ with $a \in U_a$ for every $a \in X$, and restrict it to an open ball around $\varphi_a(a) \in V_a$ in the target. These new charts will then still cover $X$, and thus they suffice to determine the smooth structure. In other words, we can assume that all charts have targets which are open balls in $\mathbb{R}^n$. By composing with a scalar multiplication and a translation, we might even assume that the target spaces of all charts are the open unit ball in $\mathbb{R}^n$.

(b) In fact, the open unit ball $U_1(0) \subset \mathbb{R}^n$ is diffeomorphic to all of $\mathbb{R}^n$ by the smooth map

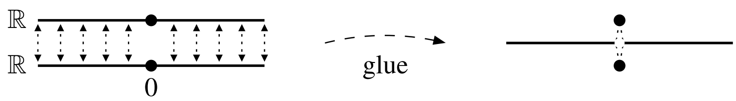
$$\varphi\colon U_1(0) \to \mathbb{R}^n,\ x \mapsto \frac{x}{\sqrt{1-\|x\|^2}} \quad \text{with inverse} \quad \varphi^{-1}\colon \mathbb{R}^n \to U_1(0),\ x \mapsto \frac{x}{\sqrt{1+\|x\|^2}},$$

so by composing charts as in (a) with $\varphi$ we might even assume that all charts in an atlas determining the smooth structure are homeomorphisms whose target space is all of $\mathbb{R}^n$.

The second problem with defining a manifold as a topological space with a (maximal) atlas is simply that this would still allow some pathological examples that we rather do not want to admit. We therefore need two more technical conditions from point-set topology to make our spaces well-behaved (see e. g. Proposition 11.13 and Proposition 11.16): the Hausdorff property and the second countability axiom.

**Remark 10.11** (Hausdorff spaces)**.** Recall that a topological space $X$ is called a **Hausdorff space** if points can be separated by open sets, i. e. if for any two distinct points $a,b \in X$ there are open neighborhoods $U$ of $a$ and $V$ of $b$ in $X$ with $U \cap V = \emptyset$. Almost by definition, the Hausdorff property means that limits of sequences in $X$ or functions to $X$ are unique if they exist [G3, Remark 4.6]. It is probably not hard to believe that analysis would break down badly without this property, so let us just assume this.

A standard counterexample (that we thus do not want to consider) is the "real line with a doubled origin" shown in the picture below on the right, which can be obtained by gluing two copies of $\mathbb{R}$ along the open subset $\mathbb{R}\backslash\{0\}$. This space admits a smooth structure determined by the two charts coming from the two copies of $\mathbb{R}$, but it is not a Hausdorff space as any two neighborhoods of the two origins intersect. Indeed, the sequence $(\frac{1}{n})_{n \in \mathbb{N}_{>0}}$ in this space converges to both origins simultaneously.




The main properties of Hausdorff spaces that we will need are the following:

(a) Every metric space (hence in particular every subset of $\mathbb{R}^n$) is a Hausdorff space, as well as every subset of a Hausdorff space [G3, Example 4.3]. In fact, all topological spaces occurring in these notes are easily seen to be Hausdorff spaces, so we will usually not mention this in what follows.

(b) Every compact subset of a Hausdorff space is closed [G3, Proposition 4.12].

**Exercise 10.12.** Show for any connected Hausdorff space $X$ with an atlas:

(a) $X$ is path-connected.

(b) For any two points $x,y \in X$ there is a homeomorphism $f\colon X \to X$ with $f(x) = y$.

**Definition 10.13** (Second countability)**.** A topological space $X$ is said to be **second countable** (or to satisfy the **second countability axiom**) if it has a countable basis in the sense of [G3, Definition 1.12], i. e. if there is a countable family $(U_i)_{i\in I}$ of open subsets of $X$ such that every open subset of $X$ is of the form $\bigcup_{i\in J} U_i$ for some $J \subset I$.

**Example 10.14.**

(a) $\mathbb{R}^n$ is second countable for all $n \in \mathbb{N}$: It is easy to check that a countable basis is given by all open balls $U_\varepsilon(x)$ for $\varepsilon \in \mathbb{Q}_{>0}$ and $x \in \mathbb{Q}^n$. 16

(b) Every subspace $Y$ of a second countable topological space $X$ is second countable: If $(U_i)_{i\in I}$ is a countable basis for $X$, then by definition of the subspace topology $(U_i \cap Y)_{i\in I}$ is a countable basis for $Y$.

(c) If a topological space $X$ has a countable open cover by second countable spaces, then $X$ itself is second countable: We can just take the union of countable bases for each of the open subsets to obtain a basis for the topology of $X$.

Taking (a), (b) and (c) together will readily imply that all topological spaces considered in these notes are second countable, hence we will usually not mention this any more in the future. However, it is also easy to come up with examples that do not satisfy this condition:

(d) $\mathbb{R}$ with the discrete topology is not second countable, since the only possible basis for this topology is the family of all singletons $\{x\}$ with $x \in \mathbb{R}$, which is not countable.

**Remark 10.15** (First countability)**.** In case you wonder why the condition of Definition 10.13 is called *second* countable: There is also a weaker notion of a *first countable* topological space $X$ only requiring for any point $a \in X$ a countable family $(U_i)_{i\in I}$ of open neighborhoods of $a$ such that every open neighborhood of $a$ is of the form $\bigcup_{i\in J} U_i$ for some $J \subset I$. Hence, first countability only requires locally that $X$ is not "too big", whereas second countability also requires this globally.

An example of a topological space that is not even first countable is the cocountable topology on $\mathbb{R}$, i. e. where a non-empty subset $U \subset \mathbb{R}$ is open if and only if it is the complement of a countable set [G3, Example 1.5 (c)]: Assuming that this space had a countable family $(U_i)_{i\in I}$ of open neighborhoods of 0, their intersection $\bigcap_{i\in I} U_i$ would still be uncountable. So if we pick any non-zero $x \in \bigcap_{i\in I} U_i$ then $\mathbb{R}\backslash\{x\}$ would be an open neighborhood of 0 that is not a union of open subsets in the given family.

Finally, we are now ready to define manifolds, which will be our main objects of interest for the rest of this class.

**Definition 10.16** (Manifolds)**.**

(a) For $n \in \mathbb{N}$, an $n$-dimensional (smooth) **manifold** is a second countable Hausdorff space $X \neq \emptyset$ together with a smooth $n$-dimensional structure $(\varphi_i)_{i\in I}$. We will denote it by $(X,(\varphi_i)_{i\in I})$, or just by $X$ if the smooth structure is clear from the context. Its dimension will be written as $\dim X = n$.

(b) A **morphism** between two manifolds $X$ and $Y$ is just a smooth map $f\colon X \to Y$ in the sense of Construction 10.5.

This makes manifolds into a category, which we denote by $\mathsf{Man}$.

**Example 10.17.**

(a) The topological spaces with atlases from Example 10.4 (a), (b), and (c) are manifolds: any open subset of $\mathbb{R}^n$, the sphere $S^n$ and the real projective space $\mathbb{P}^n_{\mathbb{R}}$ are $n$-dimensional manifolds for all $n \in \mathbb{N}$, and the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ is a $2n$-dimensional manifold.

In particular, we can see already that manifolds may or may not be compact, and they may or may not come with a natural embedding in a real vector space.

(b) Any non-empty open subset $U$ of an $n$-dimensional manifold $X$ is again an $n$-dimensional manifold, by restricting the charts $\varphi_i\colon U_i \to V_i$ to $U_i \cap U$ for all $i \in I$. In the following, we will always consider open subsets of a manifold as manifolds in this way.

(c) Manifolds form one of the categories in which bijective morphisms need not be isomorphisms: The morphism $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^3$ is bijective, but its inverse $f^{-1}\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \sqrt[3]{x}$ is not differentiable, and thus not a morphism.

**Remark 10.18.**

(a) Let $X$ be an $n$-dimensional manifold, and let $\varphi\colon U \to V$ be a chart with open subsets $U \subset X$ and $V \subset \mathbb{R}^n$. Reinterpreting this situation, we can then also consider $\varphi$ as a diffeomorphism between the manifolds $U$ (in the sense of Example 10.17 (b)) and $V$ (in the sense of Example 10.17 (a)): The statement to check here is just that $\varphi \circ \varphi^{-1} = \mathrm{id}_V$ is a diffeomorphism, which is obvious.

In other words, we see that *every $n$-dimensional manifold is locally isomorphic to an open subset of $\mathbb{R}^n$* – which is the exact formulation of our original idea that a manifold should be a space that "locally looks like $\mathbb{R}^n$". In practice, this means that for any *local* property of a manifold we may assume that we are talking about an open subset of $\mathbb{R}^n$. We will often use this fact in the future to simplify our arguments.

(b) By definition, every manifold is a topological space, so there is a natural functor $\mathsf{Man} \to \mathsf{Top}$ that just forgets the additional conditions and the smooth structure. In particular, every manifold $X$ has functorial singular homology and cohomology groups $H_p(X)$ resp. $H^p(X)$ for all $p \in \mathbb{Z}$.

(c) Having defined manifolds, we should probably mention the famous *Poincaré Conjecture*, which was one of the Millennium Problems with a one million dollar prize money, and has been solved by Grigori Perelman in 2010 in the positive [C]. It states that the only compact 3-dimensional topological manifold (in the sense of Remark 10.3 (b)) that is simply connected (i. e. path-connected with trivial fundamental group) is the sphere $S^3$.

Obviously, we are not going to prove this statement in these notes. For us, the Poincaré Conjecture should just show that even seemingly simple questions about the geometric shape of manifolds can be very hard and are still problems of current research.

**Construction 10.19** (Disjoint unions and products of manifolds)**.** Let $(X,(\varphi_i)_{i\in I})$ with $\varphi_i \colon U_i \to V_i$ and $(Y,(\psi_j)_{j\in J})$ with $\psi_j \colon U'_j \to V'_j$ be two manifolds of dimensions $n = \dim X$ and $m = \dim Y$. The following constructions then work without any problems:

(a) If $m = n$ the *disjoint union* of $X$ and $Y$ is again an $n$-dimensional manifold: We can just take the charts of both parts together, and there are no compatibility issues since charts on $X$ and $Y$ will never intersect.

Note that this really requires $m = n$: By definition, a manifold does not have to be connected, but all its charts must have the same dimension.

(b) Taking the product maps $\varphi_i \times \psi_j \colon U_i \times U'_j \to V_i \times V'_j \subset \mathbb{R}^{n+m}$ for all pairs $(i,j) \in I \times J$, the *product* $X \times Y$ is a manifold of dimension $n+m$ in a natural way. In particular, the torus $S^1 \times S^1$ is a 2-dimensional manifold.