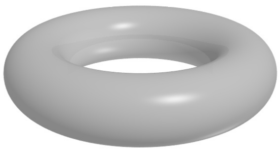
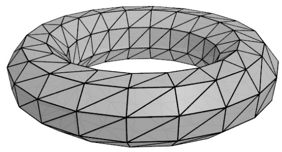
Simplicial Homology

1. Simplicial Complexes

As we mentioned already in the introduction, the goal of this course is the global study of topological spaces, or intuitively, to define and search for "holes" in them. However, in order to keep things simple for the beginning, we will restrict our attention in the first part of these notes up to Chapter 5 to topological spaces called *simplicial complexes* that can be described entirely combinatorially. Intuitively, this means for example that a torus will for the moment not look for us like in the following picture on the left, but rather as in the picture on the right – obtained from triangles by gluing them along their edges.

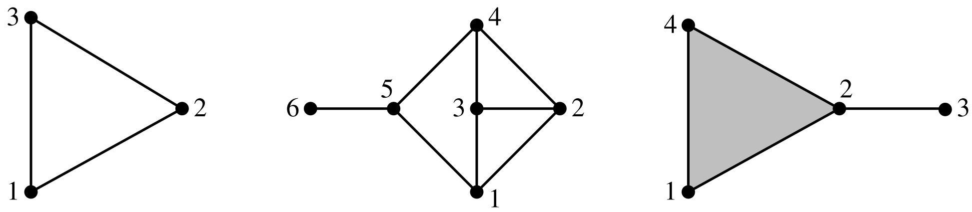




Another important goal of this first part of the notes is to introduce all the algebraic concepts from homological algebra that we will need throughout the course. In fact, in many cases simplicial complexes can really be thought of as a geometric interpretation of homological algebra, so they will also help us to understand the idea behind these algebraic concepts.

In this first chapter, we will just construct simplicial complexes rigorously and study some basic constructions with them. The question of how to find holes in them will then be addressed in Chapter 2.

Remark 1.1 (Graphs)**.** In your (mathematical) life you have probably come across *graphs*. Intuitively, graphs are finite combinatorial structures that are made up from *vertices* (sometimes called *nodes* or just *points*) and *edges* connecting them. In the following picture, S_1 and S_2 are examples of graphs, with the dots representing the vertices and the lines representing the edges.



S_1 S_2 S_3

Formally, a graph can be given by assigning labels to the vertices as in the picture, and considering each edge as a two-element set whose elements are just the labels of the connected vertices. For our purposes it will be convenient to add the vertices themselves as one-element sets to the structure, so that we could e. g. write the graph S_1 above as $S_1 = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$.

Simplicial complexes generalize this idea in the sense that we are also allowed to connect more than two vertices, e. g. to connect three vertices by a "triangle" as for S_3 in the picture above.

Definition 1.2 (Abstract simplices and simplicial complexes)**.**

(a) An **(abstract) simplex** is just a non-empty finite set α. Any simplex β with $\beta \subset \alpha$ will be called a **face** of α; it is a **proper face** if $\beta \subsetneq \alpha$.

(b) An **(abstract) simplicial complex** is a finite set S of simplices that is closed under taking faces, i. e. such that for all simplices $\beta \subset \alpha$ with $\alpha \in S$ we have $\beta \in S$.

(c) Any element of a simplex α is called a **vertex** of α. Any element of a simplex in a simplicial complex S is called a vertex of S. We denote the set of all vertices of S by $V(S)$.

(d) If a simplex α has exactly $p+1$ vertices we say that the **dimension** $\dim \alpha$ of α is p, and that α is a p**-simplex**. For a simplicial complex S, its **dimension** $\dim S$ is defined to be the maximum dimension of its simplices. A 1-simplex is called an **edge**, and a simplicial complex of dimension at most 1 is called a **graph**.

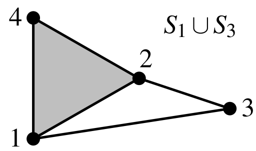
Example 1.3. Let us reconsider S_1, S_2, and S_3 as in Remark 1.1.

(a) S_1, S_2, and S_3 are simplicial complexes. While S_1 and S_2 are graphs, the simplicial complex S_3 is 2-dimensional due to its 2-simplex $\{1,2,4\} \in S_3$ with the three vertices 1, 2, and 4. The vertex set of S_3 is $V(S_3) = \{1,2,3,4\}$.

(b) Clearly, intersections and unions of simplicial complexes are again simplicial complexes. For example, we have

$$S_1 \cap S_3 = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\}\},$$

and the union $S_1 \cup S_3$ is shown in the picture on the right.



Definition 1.4 (Simplicial subcomplexes)**.** A **(simplicial) subcomplex** of a simplicial complex S is just a simplicial complex $T \subset S$. We say that it is **full** if for any $\alpha \in S$ with $\alpha \subset V(T)$ we have $\alpha \in T$.

Remark 1.5. By definition, a subcomplex $T \subset S$ being full means that a simplex in S is a simplex in T if and only if all its vertices lie in T. Of course, this makes dealing with such subcomplexes much easier since they can be specified by giving just their vertices. But we will also see e. g. in Proposition 1.24 that full subcomplexes have nicer properties than arbitrary ones.

Example 1.6.

(a) For the simplicial complexes from Remark 1.1, S_1 is a full subcomplex of S_2, but not a subcomplex of S_3 (since the edge $\{1,3\}$ is in S_1, but not in S_3). The subcomplex S_3 of $S_1 \cup S_3$ (shown in Example 1.3 (b)) is not full since the edge $\{1,3\} \in S_1 \cup S_3$ is a subset of $V(S_3)$, but does not lie in S_3.

(b) The intersection $T_1 \cap T_2$ of two full subcomplexes T_1 and T_2 of a simplicial complex S is again a full subcomplex. However, the subcomplex $T_1 \cup T_2$ need not be full: $\{\{1\}\}$ and $\{\{2\}\}$ are full subcomplexes of $\{\{1\},\{2\},\{1,2\}\}$, but their union $\{\{1\},\{2\}\}$ is not.

As a first property of simplicial complexes, let us study the intuitive notion of connectedness.

Definition 1.7 (Connectedness)**.** For a non-empty simplicial complex S, the equivalence classes of the equivalence relation

$$\alpha \sim \alpha' \quad :\Leftrightarrow \quad \text{there are } \alpha_0,\dots,\alpha_n \in S \text{ with } n \in \mathbb{N} \text{ such that } \alpha = \alpha_0,\ \alpha' = \alpha_n, \\ \text{and } \alpha_{k-1} \cap \alpha_k \neq \emptyset \text{ for all } k = 1,\dots,n$$

on S are called the **connected components** of S. We say that S is **connected** if it has only one connected component.

Clearly, S is then the disjoint union of all its connected components; let us now check that these components are in fact full subcomplexes. Moreover, as $\{i\} \in S$ for every vertex $i \in V(S)$, every vertex is a vertex of a unique connected component. We will see now that on vertices the equivalence relation of Definition 1.7 is completely analogous to the notion of path-connectedness of a topological space.

Lemma 1.8. *Let S be a simplicial complex.*

(a) *Every connected component of S is a full subcomplex of S.*

(b) *Two vertices $i, i' \in V(S)$ are vertices of the same connected component of S if and only if they can be connected by a chain of edges in S, i. e. if there are $n \in \mathbb{N}$ and edges $\alpha_0, \dots, \alpha_n$ with $i \in \alpha_0$, $i' \in \alpha_n$, and $\alpha_{k-1} \cap \alpha_k \neq \emptyset$ for all $k = 1, \dots, n$.*

Proof.

(a) Let $T \subset S$ be a connected component, i. e. a class of the equivalence relation of Definition 1.7. As $\beta \sim \alpha$ for every face β of α it is clear that T is a subcomplex. To show that it is full, let $\alpha = \{i_0, \dots, i_p\} \in S$ be a p-simplex with $i_0, \dots, i_p \in V(T)$. Then $\alpha \sim \{i_0\} \in T$, so it follows that $\alpha \in T$.

(b) The direction "$\Leftarrow$" is clear from the definition. For the direction "$\Rightarrow$" assume that i and i' are two vertices of the same connected component, i. e. that there are $\alpha_0, \dots, \alpha_n \in S$ with $\{i\} = \alpha_0$, $\{i'\} = \alpha_n$, and $\alpha_{k-1} \cap \alpha_k \neq \emptyset$ for all $k = 1, \dots, n$. Pick a vertex $i_k \in \alpha_{k-1} \cap \alpha_k$ for all $k = 1, \dots, n$. Then $i = i_1$, $i' = i_n$, and $\{i_{k-1}, i_k\} \subset \alpha_{k-1} \in S$ for all $k = 2, \dots, n$, so $\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{n-1}, i_n\}$ gives the desired chain. □

Example 1.9. Relabeling the vertices $1, 2, 3, 4$ in the simplicial complex S_3 of Remark 1.1 to $1', 2', 3', 4'$ to obtain the simplicial complex S_3' makes $S_1 \cup S_3'$ into a disconnected simplicial complex with connected components S_1 and S_3'.

So far we have considered a simplicial complex S only as a combinatorial structure. But for our purposes we also want to realize it as a topological space, e. g. to interpret the pictures in Remark 1.1 as topological subspaces of $\mathbb{R}^2$ obtained by picking some points for the vertices of S and joining them by lines or triangles according to the simplices in S. In arbitrary dimensions, these building blocks will be called *geometric simplices*. Of course, for this idea to work we have to pick the points for the vertices general enough so that e. g. three points are not on a line if we want to join them by a triangle and two lines do not intersect if they do not have a vertex in common. Let us quickly go through the constructions necessary to achieve this.

Definition 1.10 (Affine independence)**.** We say that finitely many points $v_0, \dots, v_p \in W$ in a real vector space W are called **affinely independent** if for all $x_0, \dots, x_p \in \mathbb{R}$ with $x_0 v_0 + \cdots + x_p v_p = 0$ and $x_0 + \cdots + x_p = 0$ we have $x_0 = \cdots = x_p = 0$.

Remark 1.11.

(a) Using that $x_0 = -x_1 - \cdots - x_p$ we can rephrase the condition of Definition 1.10 by saying that $x_1(v_1 - v_0) + \cdots + x_p(v_p - v_0) = 0$ with $x_1, \dots, x_p \in \mathbb{R}$ implies $x_1 = \cdots = x_p = 0$. Hence the affine independence of $v_0, \dots, v_p$ just means that shifting one of the vectors to the origin makes the other ones linearly independent, or in other words that $v_0, \dots, v_p$ span a shifted p-dimensional linear subspace.

(b) Obviously, linear independence implies affine independence, but not vice versa (as e. g. 1 and 2 are affinely, but not linearly independent in $\mathbb{R}$).

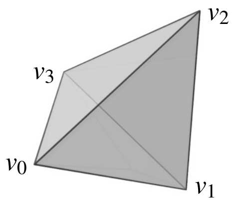
Construction 1.12 (Geometric simplices and barycentric coordinates)**.** Let $v_0, \dots, v_p \in W$ be points in a real vector space W.

(a) We denote the convex hull spanned by $v_0, \dots, v_p$ by

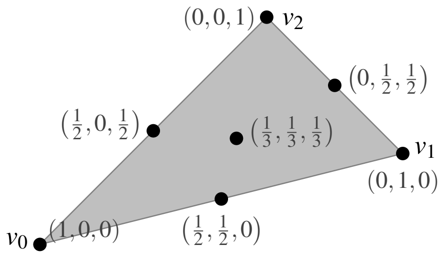
$$\langle v_0, \dots, v_p \rangle := \{x_0 v_0 + \cdots + x_p v_p : x_0, \dots, x_p \in \mathbb{R}_{\geq 0} \text{ with } x_0 + \cdots + x_p = 1\} \quad \subset W.$$

If $v_0, \dots, v_p$ are affinely independent, we call this set the **geometric p-simplex** spanned by $v_0, \dots, v_p$. A **face** of it is any geometric simplex spanned by a subset of the given points, and it is a **proper face** if it is spanned by a proper subset. We call the union of all proper faces the **boundary** and its complement in the geometric simplex the **interior** of $\langle v_0, \dots, v_p \rangle$.

For example, a geometric 3-simplex $\langle v_0, v_1, v_2, v_3 \rangle$ is a tetrahedron as in the following picture on the left. Two of its proper faces are the geometric 0-simplex $\langle v_0 \rangle = \{v_0\}$ (which is just a point) and the geometric 2-simplex $\langle v_0, v_1, v_2 \rangle$ (which is a triangle).



$\langle v_0, v_1, v_2, v_3 \rangle$



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(b) Note that in (a) the values of $x_0, \dots, x_p$ leading to a point $x = x_0 v_0 + \cdots + x_p v_p$ in the geometric p-simplex spanned by $v_0, \dots, v_p$ are uniquely determined: For another representation $x = x'_0 v_0 + \cdots + x'_p v_p$ with $x'_0 + \cdots + x'_p = 1$ we obtain

$$(x_0 - x'_0) v_0 + \cdots + (x_p - x'_p) v_p = 0 \quad \Rightarrow \quad x_i = x'_i \text{ for all } i$$

since $\sum_{i=0}^p (x_i - x'_i) = \sum_{i=0}^p x_i - \sum_{i=0}^p x'_i = 1 - 1 = 0$ and $v_0, \dots, v_p$ are affinely independent. We call $(x_0, \dots, x_p)$ the **barycentric coordinates** of x (the name comes from the word "barycenter" meaning "center of mass", and in fact x is by definition the weighted average of $v_0, \dots, v_p$ with the weights $x_0, \dots, x_p$).

In summary, barycentric coordinates of a point in $\langle v_0, \dots, v_p \rangle$ are uniquely defined, non-negative, and sum up to 1. The interior of $\langle v_0, \dots, v_p \rangle$ consists exactly of the points whose barycentric coordinates are all positive. The picture above on the right shows some points in the triangle $\langle v_0, v_1, v_2 \rangle$ with their barycentric coordinates (x_0, x_1, x_2).

Let us now assume that we are given a simplicial complex S and a "realization map" $r \colon V(S) \to W$ to a real vector space that tells us at which points we want the vertices to be. We can then try to connect these points by simplices according to S to obtain a geometric realization of S.

Construction 1.13 (Geometric realizations)**.** For a simplicial complex S and a map $r \colon V(S) \to W$ to a real vector space W we set

$$r(\alpha) := \langle r(i_0), \dots, r(i_p) \rangle \quad \subset W \quad \text{for any } p\text{-simplex } \alpha = \{i_0, \dots, i_p\} \in S, \text{ with } r(\emptyset) := \emptyset$$

$$\text{and} \quad r(S) := \bigcup_{\alpha \in S} r(\alpha) \quad \subset W.$$

We call r a **geometric realization** of S if $r(\alpha) \cap r(\beta) = r(\alpha \cap \beta)$ for any $\alpha, \beta \in S$. By abuse of notation, we will then often also say that $r(S)$ is the geometric realization of S determined by r.

Remark 1.14 (Topological realizations)**.** Let $r \colon V(S) \to W$ be a geometric realization of a simplicial complex S.

(a) Clearly, $r(\beta)$ is a face of $r(\alpha)$ for any two simplices $\beta \subset \alpha$ in S.

(b) If $\alpha \in S$ is an abstract p-simplex then $r(\alpha)$ is a geometric p-simplex, i. e. the points $r(i)$ for $i \in \alpha$ are affinely independent: Assume for a contradiction that we have coordinates $x_i \in \mathbb{R}$ for $i \in \alpha$ with $\sum_{i \in \alpha} x_i \, r(i) = 0$ and $\sum_{i \in \alpha} x_i = 0$, where at least one x_i is non-zero. For $\alpha_+ := \{i \in \alpha : x_i > 0\}$ and $\alpha_- := \{i \in \alpha : x_i < 0\}$ we then have

$$x := \sum_{i \in \alpha_+} x_i \, r(i) = \sum_{i \in \alpha_-} (-x_i) \, r(i).$$

The coordinates on both sides are now positive and sum up to the same number due to the condition $\sum_{i \in \alpha} x_i = 0$, so by normalizing we may assume that the sum of the coordinates is 1 on both sides. But then it follows that $x \in r(\alpha_+) \cap r(\alpha_-) = r(\alpha_+ \cap \alpha_-) = r(\emptyset) = \emptyset$, which is a contradiction.

(c) It follows by (a) and (b) that the geometric realization $r(S)$ can also be constructed *as a topological space* by a gluing process: We can start with the *disjoint* union of all geometric simplices $r(\alpha)$ for $\alpha \in S$, and then glue any two geometric simplices $r(\alpha)$ and $r(\beta)$ along their common face $r(\alpha \cap \beta)$, giving the resulting space the quotient topology [G3, Chapter

5]. Note that this captures precisely the idea how we wanted to consider abstract simplicial complexes as topological spaces.

Moreover, as any two p-simplices are homeomorphic this means that any two geometric realizations of S are homeomorphic. If we think of $r(S)$ this way we will also refer to it as the **topological realization** of S.

It just remains to be shown that every simplicial complex admits a geometric realization at all. In fact, we will now show that it even has a canonically defined one.

Proposition 1.15 (Standard geometric realization)**.** *Let S be a simplicial complex. Then the map $r\colon V(S) \to \mathbb{R}^{V(S)}$, $i \mapsto e_i$ is a geometric realization of S. We call it the* ***standard geometric realization*** *of S, and denote its images in $R^{V(S)}$ by $|\alpha| := r(\alpha)$ and $|S| := r(S)$ for a simplex $\alpha \in S$.*

Proof. Let α and β be two simplices of S; we have to show that $r(\alpha) \cap r(\beta) = r(\alpha \cap \beta)$.

"$\subset$": Let $x \in r(\alpha) \cap r(\beta)$, so there are barycentric coordinates x_i for $i \in \alpha$ and x_i' for $i \in \beta$ with $x = \sum_{i\in\alpha} x_i e_i = \sum_{i\in\beta} x_i' e_i$. As all unit vectors are independent this means that x_i and x_i' can only be non-zero for $i \in \alpha \cap \beta$; hence $x \in r(\alpha \cap \beta)$.

"$\supset$": This follows immediately from Remark 1.14 (a) as $\alpha \cap \beta$ is a face of both α and β. □

Remark 1.16. The standard geometric realization $|S|$ of a simplicial complex S has the disadvantage of having a high-dimensional target space, so that it is usually not useful for visualization. On the other hand, it has several advantages:

(a) It does not depend on any choices.

(b) It has barycentric coordinates that are defined globally and not just on a fixed simplex: By construction, we have

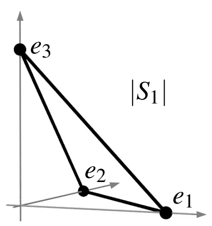
$$|S| = \left\{ x \in \mathbb{R}^{V(S)}_{\geq 0} : \sum_{i \in V(S)} x_i = 1 \text{ and } \{i \in V(S) : x_i > 0\} \in S \right\} \quad \subset \mathbb{R}^{V(S)}.$$

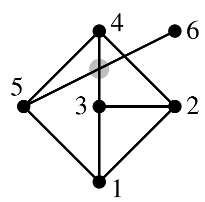
(c) Any other geometric realization $r\colon V(S) \to W$ is obtained from it by a linear map that is injective on $|S|$, namely by the linear map $\mathbb{R}^{V(S)} \to W$ determined by $e_i \mapsto r(i)$ for all $i \in V(S)$.

Hence, for topological purposes (when the realization does not matter by Remark 1.14 (c)), we will usually use the standard geometric realization.

Example 1.17. Consider again the simplicial complexes S_1, S_2, and S_3 of Remark 1.1.

(a) The picture below on the left shows the standard geometric realization $|S_1| \subset \mathbb{R}^{\{1,2,3\}}$ of S_1. All pictures in Remark 1.1 show valid geometric realizations of the given simplicial complexes when we consider them as subsets of $\mathbb{R}^2$, with the realization map r given by mapping each vertex i to the point labeled i in the picture. In contrast, the map $r\colon V(S_2) \to \mathbb{R}^2$ shown below on the right is not a geometric realization of S_2 since $r(\{3,4\}) \cap r(\{5,6\})$ contains the gray point, whereas $r(\{3,4\} \cap \{5,6\}) = r(\emptyset) = \emptyset$.





(b) Let α be a p-simplex. Then α together with all of its faces, i. e.

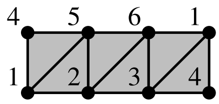
$$D^\alpha := \{\beta : \emptyset \neq \beta \subset \alpha\}$$

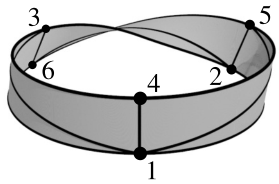
is a simplicial complex whose topological realization is a p-dimensional disk D^p. Similarly, for $p > 0$ the set of all proper faces

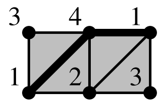
$$S^\alpha := \{\beta : \emptyset \neq \beta \subsetneq \alpha\}$$

is a simplicial complex whose topological realization is the sphere S^{p-1}. For example, the simplicial complex S_1 is just $S^{\{1,2,3\}}$.

(c) To construct a topological realization it is sometimes useful to only use the gluing construction of Remark 1.14 (c) without any embedding in a real vector space. For example, the **Möbius strip** is the topological space obtained from the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$ by gluing the left to the right side in opposite directions, giving the resulting space the quotient topology [G3, Example 5.9 (b)]. It is visibly the topological realization of the simplicial complex shown in the picture below on the left, where the identification of the left and right side in opposite directions is implied by the numbering. Note that it is not a subspace of $\mathbb{R}^2$, but can be visualized as a topological space (not as a geometric realization!) in $\mathbb{R}^3$ as in the middle picture.

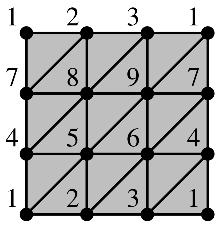




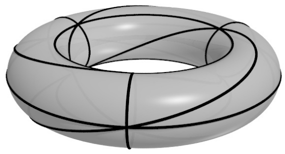


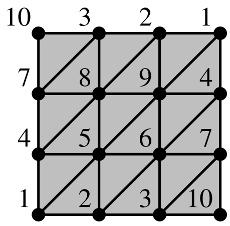
Also note that subdividing the unit square horizontally into only two instead of three parts as in the right picture would not have been sufficient: This is not a valid simplicial complex as the vertices 1 and 4 are joined by two edges (drawn in bold) – which is not allowed since a simplicial complex is just a set and cannot contain a simplex multiple times.

(d) Similarly to (b), we can obtain a **torus** by taking a unit square and identifying the left with the right and the top with the bottom side in the same direction [G3, Example 5.9 (c)]. A simplicial complex achieving this is shown in the picture below on the left; it consists of 9 zero-dimensional, 27 one-dimensional, and 18 two-dimensional simplices. As a topological space this is just the product $S^1 \times S^1$ of two circles in the middle picture.



Torus





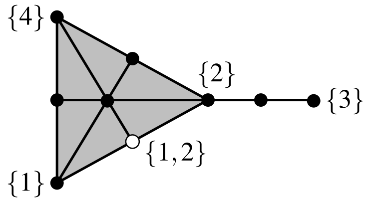
Real projective plane

In contrast, the picture above on the right shows a simplicial complex realizing the topological space obtained from gluing opposite sides of the unit square in opposite directions. Note that it needs one vertex more since now not all four corners of the square are identified, but only opposite ones. Although this simplicial complex might look very similar to that for the torus, it gives a fundamentally different topological space called the **real projective plane** [G3, Example 5.9 (c)]. We will consider it in more detail later in Construction 8.3.

Exercise 1.18. Draw an abstract simplicial complex whose topological realization is the real projective plane and that has fewer than 10 vertices.

Exercise 1.19. Let $T_1, \dots, T_n$ be the connected components of a simplicial complex S. Show that $|T_1|, \dots, |T_n|$ are the path-connected components of the topological space $|S|$.

Of course, it may happen that different simplicial complexes have homeomorphic topological realizations. The most obvious way to achieve this is by subdividing the simplices in a simplicial complex S. A symmetric approach to this construction is to add a vertex in the barycenter (i. e. midpoint) of each simplex $\alpha \in S$ and subdivide accordingly. For example, after this subdivision process the simplicial complex S_3 of Remark 1.1 would look like the picture on the right, with the hollow vertex being the barycenter of the edge $\{1,2\}$.



Note that in this way the vertex set of the subdivided complex can be identified with the set S of simplices in the original simplicial complex. In doing so, the old vertices $i \in V(S)$ will be relabeled by the 0-simplices $\{i\} \in S$ as in the picture above. Let us now define this subdivision process rigorously.

Definition 1.20 (Barycentric subdivision)**.** For a simplicial complex S, the simplicial complex

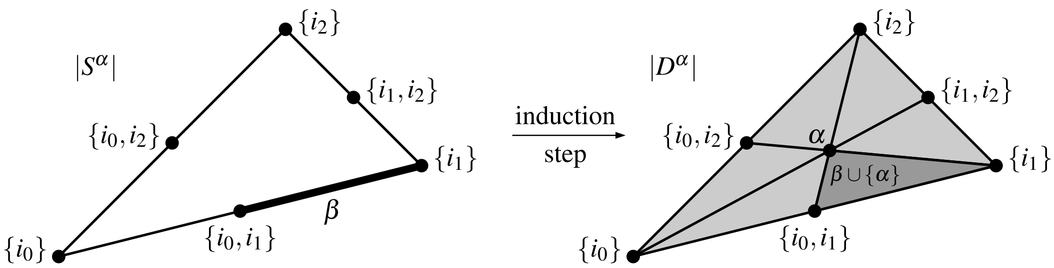
$$\mathrm{Sd}(S) := \big\{\{\alpha_0,\dots,\alpha_p\} : p \in \mathbb{N} \text{ and } \emptyset \neq \alpha_0 \subsetneq \cdots \subsetneq \alpha_p \in S\big\}$$

with vertex set $V(\mathrm{Sd}(S)) = S$ is called the **barycentric subdivision** of S.

Proposition 1.21 (Barycentric realization)**.** *Let S be an abstract simplicial complex. Then the map $r\colon V(\mathrm{Sd}(S)) \to \mathbb{R}^{V(S)}$ that sends each p-simplex $\alpha = \{i_0,\dots,i_p\} \in S$ to its barycenter $r(\alpha) := \frac{1}{p+1}(e_{i_0} + \cdots + e_{i_p})$ is a geometric realization of* $\mathrm{Sd}(S)$ *with* $r(\mathrm{Sd}(S)) = |S|$. *We call it the **barycentric realization** of* $\mathrm{Sd}(S)$.

Sketch proof. We show the statement by induction on the cardinality of S, with the case $\dim S = 0$ being obvious. So assume that $\dim S > 0$, choose a simplex $\alpha = \{i_0,\dots,i_p\}$ of maximum dimension p in S, and set $S' := S \backslash \{\alpha\}$. By induction, we know that r is a geometric realization of $\mathrm{Sd}(S')$ with $r(\mathrm{Sd}(S')) = |S'|$.

Note that all proper faces of α must already be in S'. So S' contains the sphere S^α, and hence $\mathrm{Sd}(S')$ contains the subdivided sphere $\mathrm{Sd}(S^\alpha)$, which again by induction is realized by r with image $|S^\alpha|$. A picture of this situation in the case $p = 2$ is shown below on the left (where the simplices in $\mathrm{Sd}(S') \backslash \mathrm{Sd}(S^\alpha)$ are not shown for simplicity).



By construction, the transition from S' to S now just adds the simplex α, hence by Definition 1.20 the transition from $\mathrm{Sd}(S')$ to $\mathrm{Sd}(S)$ adds the simplices $\{\alpha_0,\dots,\alpha_k\}$ for $k \in \mathbb{N}$ and $\emptyset \neq \alpha_0 \subsetneq \cdots \subsetneq \alpha_k = \alpha$. This is precisely the 0-simplex $\{\alpha\}$ together with all simplices of the form $\beta \cup \{\alpha\}$ for $\beta \in \mathrm{Sd}(S^\alpha)$. As in the picture above on the right, placing $\alpha \in V(\mathrm{Sd}(S))$ at its barycenter the transition from $\mathrm{Sd}(S')$ to $\mathrm{Sd}(S)$ thus fills in the sphere $|S^\alpha|$ to a disk $|D^\alpha|$. Hence r realizes $\mathrm{Sd}(S)$ with image $|S'| \cup |D^\alpha| = |S'| \cup |\alpha| = |S|$. □

Remark 1.22.

(a) As one can see from the sketch proof above, it actually does not matter that we place the vertices of the subdivision in the barycenters – any point in the interior of the corresponding geometric simplex would work equally well. Also, in the same way we could start with any (non-standard) geometric realization r of a simplicial complex S and then construct a barycentric realization of $\mathrm{Sd}(S)$ with the same image as $r(S)$.

(b) Obviously, barycentric subdivisions behave nicely with respect to subcomplexes, intersections, and unions: If T is a subcomplex of a simplicial complex S then $\mathrm{Sd}(T)$ is a subcomplex of $\mathrm{Sd}(S)$; and if S and T are arbitrary simplicial complexes then $\mathrm{Sd}(S \cap T) = \mathrm{Sd}(S) \cap \mathrm{Sd}(T)$ and $\mathrm{Sd}(S \cup T) = \mathrm{Sd}(S) \cup \mathrm{Sd}(T)$.

Barycentric subdivisions will come up at various places, but to see their usefulness now let us prove as a first result that they can be used to make subcomplexes full, and to construct open neighborhoods of subcomplexes with nice properties.

Lemma 1.23. *If T is a subcomplex of a simplicial complex S, then $\mathrm{Sd}(T)$ is a full subcomplex of $\mathrm{Sd}(S)$.*

Proof. By definition, $\mathrm{Sd}(T)$ is clearly a simplicial subcomplex of $\mathrm{Sd}(S)$. To show that it is full, let $\beta = \{\alpha_0, \dots, \alpha_p\} \in \mathrm{Sd}(S)$, i. e. $\emptyset \neq \alpha_0 \subsetneq \cdots \subsetneq \alpha_p \in S$, such that $\alpha_0, \dots, \alpha_p \in V(\mathrm{Sd}(T)) = T$. Then $\emptyset \neq \alpha_0 \subsetneq \cdots \subsetneq \alpha_p \in T$, and hence $\beta \in \mathrm{Sd}(T)$. □

Proposition 1.24 (Standard neighborhoods)**.** *For a subcomplex T of a simplicial complex S we set*

$$N_{T/S} := \{x \in |S| : x_i > 0 \text{ for some } i \in V(T)\}.$$

(a) *$N_{T/S}$ is an open subset of $|S|$ that contains $|T|$ (i. e. an open neighborhood of $|T|$ in $|S|$).*

(b) *If T is full then $|T|$ is a deformation retract of $N_{T/S}$. In particular, $N_{T/S}$ is then homotopy equivalent to $|T|$.*

*We call $N_{T/S}$ the **standard neighborhood** of $|T|$ in $|S|$.*

Proof.

(a) It is clear from the definition that $N_{T/S}$ is open. Moreover, every $x \in |T|$ has barycentric coordinates x_i with $i \in V(T)$ that sum up to 1. Hence, at least one of them is non-zero, and it follows that $x \in N_{T/S}$.

(b) Let us consider the map

$$\pi\colon N_{T/S} \to \mathbb{R}^{V(T)},\ x \mapsto \frac{\sum_{i \in V(T)} x_i e_i}{\sum_{i \in V(T)} x_i}$$

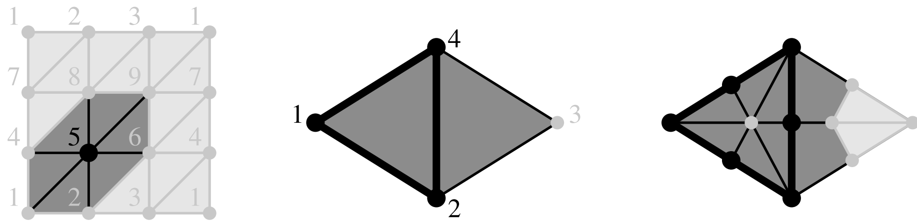
that projects to $\mathbb{R}^{V(T)} \subset \mathbb{R}^{V(S)}$ and normalizes. It is in fact a well-defined continuous map as the denominator will never be 0 on $N_{T/S}$. We claim that its image lies in $|T|$. To see this, note first that for any $x \in N_{T/S} \subset |S|$ the coordinates of $\pi(x)$ are clearly still non-negative and sum up to 1. Moreover, by definition of $|S|$ we have $x \in |\alpha|$ for some $\alpha \in S$, and thus $\pi(x) \in |\alpha \cap V(T)|$. But $\alpha \cap V(T)$ is a simplex in S by Definition 1.2 (b) and has its vertices in $V(T)$, so it is a simplex of T since T is full. It follows that $\pi(x) \in |T|$, i. e. we have constructed a continuous map $\pi\colon N_{T/S} \to |T|$. Finally, π is in fact a deformation retraction since

$$H\colon N_{T/S} \times [0,1] \to N_{T/S},\ (x,t) \mapsto \frac{\sum_{i \in V(T)} x_i e_i + t \sum_{i \notin V(T)} x_i e_i}{\sum_{i \in V(T)} x_i + t \sum_{i \notin V(T)} x_i}$$

(which moves each point of $\mathbb{R}^{V(S)}$ linearly to its projection to $\mathbb{R}^{V(T)}$ and normalizes) is a homotopy between $H(\cdot,0) = \pi$ and $H(\cdot,1) = \mathrm{id}_{N_{T/S}}$. □

Example 1.25. By Construction 1.12 (b), for a vertex i of a simplicial complex S the set of all $x \in |S|$ with $x_i > 0$ consists of all interiors of simplices containing the vertex i. This explains the shape of the standard neighborhoods $N_{T/S}$ in the picture below, where the simplicial complex S is drawn in light color, the subcomplex T in bold, and $N_{T/S}$ in dark color.

(a) Let S be the torus of Example 1.17 (d), and let T be the 0-dimensional subcomplex of S with the single vertex 5. The standard neighborhood $N_{T/S}$ is shown in the picture on the left and can be thought of as a simplicial version of an open ball neighborhood in analysis.



(b) As in the middle picture, consider the simplicial complex $S = D^{\{1,2,4\}} \cup D^{\{2,3,4\}}$ with the subcomplex $T = S^{\{1,2,4\}}$. The standard neighborhood $N_{T/S}$ is then all of $|S|$ except the vertex 3. While this clearly is a neighborhood of $|T|$, it is not homotopy equivalent to $|T|$ since it is contractible whereas the circle $|T|$ is not. However, as in the picture above on the right, if we perform a barycentric subdivision first then the subcomplex becomes full by Lemma 1.23, and consequently its geometric realization is now a deformation retract of its standard neighborhood by Proposition 1.24 (with the retraction going in the direction of the thin black lines). Intuitively, the reason for this is that the subdivision makes the simplicial structure fine enough so that the standard neighborhood does not completely fill in the hole of the circle $|T|$.

Exercise 1.26. Prove that for any two subcomplexes T_1 and T_2 of a simplicial complex S we have:

(a) $N_{(T_1 \cup T_2)/S} = N_{T_1/S} \cup N_{T_2/S}$;

(b) $N_{(T_1 \cap T_2)/S} \subset N_{T_1/S} \cap N_{T_2/S}$, with equality if $T_1 \cup T_2$ is full.

Let us conclude this chapter with a definition of morphisms of simplicial complexes, which is also completely combinatorial but captures the idea of a continuous map between the topological realizations.

Definition 1.27 (Morphisms of simplicial complexes)**.** Let S and T be simplicial complexes.

(a) A **morphism** f from S to T is a map $f\colon V(S) \to V(T)$ such that for any $\alpha \in S$ we have $f(\alpha) \in T$. We will write this as $f\colon S \to T$.

(b) Note that any such morphism $f\colon S \to T$ induces a unique continuous map from $|S|$ to $|T|$ that sends e_i to $e_{f(i)}$ for all $i \in V(S)$ and is affine-linear on every simplex of S. We will denote it by $|f|\colon |S| \to |T|$.

Example 1.28.

(a) Clearly, the identity $\mathrm{id}_S\colon S \to S$ given by $\mathrm{id}_S(i) = i$ for all $i \in V(S)$ is a morphism. More generally, for any subcomplex T of S there is an inclusion morphism $f\colon T \to S$ given by $f(i) = i$ for all $i \in V(T)$.

(b) With the labeling of the vertices as in Example 1.17 (c) and (d) there is a morphism f from the Möbius strip to the real projective plane given by $f(i) = i+3$ for all $i \in \{1,\dots,6\}$.

(c) Note that a morphism of simplicial complexes need not be injective, and it does not even have to preserve the dimension of simplices. For example, we could map the torus with the labeling as in Example 1.17 (d) to the circle $S^{\{1,2,3\}}$ by sending the vertex $3i+j$ to j for all $i = \{0,1,2\}$ and $j \in \{1,2,3\}$. This would map e. g. the 2-simplex $\{1,2,5\}$ of the torus to the edge $\{1,2,2\} = \{1,2\}$ of the circle. Topologically, this morphism just corresponds to the projection $S^1 \times S^1 \to S^1$ of the torus to one of its factors.

(d) Clearly, the composition of morphisms of simplicial complexes is again a morphism.

Definition 1.29 (Isomorphisms of simplicial complexes)**.** A morphism $f\colon S \to T$ of simplicial complexes is an **isomorphism** if there is a morphism $g\colon T \to S$ with $g \circ f = \mathrm{id}_S$ and $f \circ g = \mathrm{id}_T$. Two simplicial complexes are called **isomorphic** if there is an isomorphism between them.

Example 1.30.

(a) Relabeling the vertices of a simplicial complex produces an isomorphic simplicial complex.

(b) Of course, any isomorphism $f\colon S \to T$ of simplicial complexes induces a homeomorphism $|f|\colon |S| \to |T|$ between their topological realizations. We have seen already however that non-isomorphic simplicial complexes may have homeomorphic topological realizations, e. g. any simplicial complex and its barycentric subdivision.

In fact, in this course we will often construct certain mathematical objects (such as simplicial complexes in this chapter) and structure-preserving morphisms between them, and relate them to other objects of the same type (e. g. subdivide a simplicial complex) or of different types (e. g. associate a topological space to a simplicial complex). The first questions and properties for such constructions are always the same, so it is useful to introduce a unified language to deal with them. This is
02 achieved by the notion of *categories* and *functors* that we will now briefly explain.

Notation 1.31 (Categories)**.** A **category** C is given by specifying

- a class of objects of C; and
- for any two objects of C a class of morphisms between them, such that morphisms can be composed, composition is associative, and any object has an identity morphism.

Examples that we already know and that will be relevant for us are:

(a) the category Vect whose objects are vector spaces over a fixed ground field, and morphisms are linear maps;

(b) the category Top whose objects are topological spaces, and morphisms are continuous maps;

(c) the category Simp whose objects are simplicial complexes, and morphisms are as in Definition 1.27 (a).

In a category, as in Definition 1.29 an *isomorphism* will always be a morphism that has a two-sided inverse, and two objects will be called *isomorphic* if there is an isomorphism between them.

Notation 1.32 (Functors)**.** For two categories C and D a **(covariant) functor** $F\colon \mathsf{C} \to \mathsf{D}$ is given by:

- for every object X of C an object $F(X)$ of D;
- for every morphism $f\colon X \to Y$ of objects of C a morphism $F(f)\colon F(X) \to F(Y)$ of objects of D, such that $F(g \circ f) = F(g) \circ F(f)$ and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any composition $g \circ f$ of morphisms in C and any object X of C.

Again, we have already seen some examples of functors in this chapter:

(a) the *topological realization functor* $|\cdot|\colon \mathsf{Simp} \to \mathsf{Top}$ that sends a simplicial complex S to its topological realization $|S|$;

(b) the *barycentric subdivision functor* $\mathrm{Sd}\colon \mathsf{Simp} \to \mathsf{Simp}$ that sends a simplicial complex S to its barycentric subdivision $\mathrm{Sd}(S)$.

Note that in both cases we have just specified the functor on objects, since its action on morphisms is clear: For (a) it was given in Definition 1.27 (b), and for (b) a morphism $f\colon S \to T$ between simplicial complexes is mapped to the morphism $\mathrm{Sd}(f)\colon \mathrm{Sd}(S) \to \mathrm{Sd}(T)$ given on a vertex $\alpha \in V(\mathrm{Sd}(S)) = S$ by $\mathrm{Sd}(f)(\alpha) = f(\alpha)$. In both cases, checking the required properties is straightforward.

Sometimes, one also has assignments on objects such that the correspondence for morphisms works in the opposite direction, i. e. so that for a morphism $f\colon X \to Y$ in C we have $F(f)\colon F(Y) \to F(X)$, and consequently $F(g \circ f) = F(f) \circ F(g)$ for any composition $g \circ f$ of morphisms in C. In this case, F is called a **contravariant functor**. A well-known example for this is the dualization functor ${}^\vee\colon \mathsf{Vect} \to \mathsf{Vect}$ that sends a vector space V to its dual $V^\vee$, and a linear map $f\colon V \to W$ to its dual map $f^\vee\colon W^\vee \to V^\vee$.

There is a whole branch of mathematics called *category theory* that studies the properties of general categories. We are not going to pursue this here; instead we will just use the notations introduced above as a convenient language. For example, it is clear from the above constructions that functors send isomorphisms to isomorphisms and can be composed to give new functors, so that from now on remarks as e. g. in Example 1.30 (b) do not need special mention since they are automatic.

Remark 1.33 (Variants of simplicial complexes)**.** There are two variants of simplicial complexes that often appear in the literature, but that we do not want to consider here to keep notations simple:

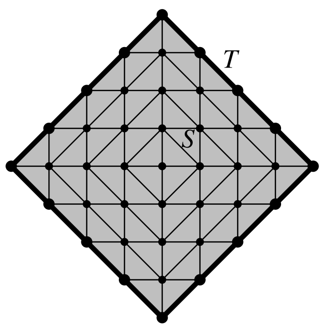
(a) In the literature, simplicial complexes are usually allowed to have infinitely many simplices, maybe even of unbounded dimension (although each simplex always has to be finite-dimensional). By allowing this, one would also be able to obtain non-compact spaces as geometric realizations, e. g. the set of simplices $\{\{n\},\{n,n+1\} : n \in \mathbb{Z}\}$ would construct the real line as an "infinite chain of edges". However, we will not do so in these notes since infinite simplicial complexes would often make computations and (inductive) proofs more complicated or even impossible. The restriction to compact geometric realizations at this point will actually turn out not to be that bad, since our final theory will be invariant under homotopy equivalence, and almost all topological spaces occurring in practice will visibly be homotopy equivalent to a compact space.

(b) By modeling simplicial complexes as sets, the vertices of each simplex have to be distinct, and it is not possible to include the same simplex in a simplicial complex multiple times. For example, in the case of graphs, we cannot connect a vertex to itself by an edge to form a loop, and we cannot connect two vertices by more than one edge (see also Example 1.17 (c)). To waive these restrictions, one can consider so-called Δ*-complexes* [H, Chapter 2.1], or *multigraphs* in the 1-dimensional case.

Exercise 1.34. Let S be the 2-dimensional simplicial complex in the picture below, with vertices labeled (i, j) for $i, j \in \mathbb{Z}$ and $|i| + |j| \leq 4$ according to their position in the plane. We denote its "boundary" by T, i. e. the full 1-dimensional subcomplex drawn in bold containing the vertices (i, j) with $|i| + |j| = 4$. Obviously, the topological realizations of S and T are the 2-dimensional disk D^2 and its boundary circle S^1, respectively.

The following two facts are known from topology:

- There is no continuous map $f: D^2 \to S^1$ with $f|_{S^1} = \mathrm{id}$.
- Every continuous map $f: D^2 \to D^2$ has a fixed point.



Are the following simplicial versions of these statements true or false? If a statement is true, give a purely simplicial proof as well as a proof deriving it from the corresponding topological fact above. If a statement is false, provide a counterexample.

(a) There is no morphism $f: S \to T$ with $f|_T = \mathrm{id}$.

(b) Every morphism $f: S \to S$ has a fixed vertex.

Exercise 1.35. Prove that every simplicial complex S admits a geometric realization in $\mathbb{R}^n$ with $n = 2\dim S + 1$.

(Hint: Start with the standard geometric realization and try to project it linearly to smaller-dimensional subspaces as long as possible. It might be useful that the complement of a finite union of proper affine subspaces of a real vector space is a dense open subset.)