0. Introduction

We all know that a donut has a hole. But what exactly does that mean, how can we prove it, and what are the consequences?

In an introductory class on topology one usually learns that the so-called fundamental group of a topological space can be used to capture this idea: Its elements are homotopy classes of maps from loops to the given space, so it measures whether every loop in the space can be contracted to a point. A donut is then shown to have a non-trivial fundamental group, and this is a precise way to phrase the existence of its hole. Moreover, you might have seen that the fundamental group can be used to prove many nice results. For example, the arguably most intuitive way to prove the fundamental theorem of algebra (i. e. that every non-constant complex polynomial has a zero) is a topological one that uses the non-contractibility of loops in $\mathbb{C}\backslash\{0\} = \mathbb{R}^2\backslash\{0\}$, and thus the fundamental group.

But not all holes are created equal: You would probably agree that the space $\mathbb{R}^3\backslash\{0\}$ has a hole in the origin as well. However, every loop in it can be contracted to a point, i. e. its fundamental group is trivial and does not see the existence of the hole. So our initial question remains, which can actually be thought of as the fundamental question of algebraic topology: What exactly does it mean that a topological space has a hole, how can we prove it, and what are the consequences?

From the example $\mathbb{R}^3\backslash\{0\}$ one can easily read off one possible idea to solve this problem: We should probably pass from loops to higher-dimensional test objects. More precisely, it seems intuitive that the unit 2-sphere in $\mathbb{R}^3\backslash\{0\}$ cannot be contracted to a point in the same way as the unit 1-sphere in $\mathbb{R}^2\backslash\{0\}$ cannot be contracted to a point. This idea would lead to the so-called higher *homotopy groups* generalizing the fundamental group to higher dimensions. Unfortunately, it turns out however that these homotopy groups are very complicated. Already the homotopy groups of spheres are surprisingly wild and hard to compute [H, Chapter 4.1].

The good news is that there are many different other and easier ways to detect higher-dimensional holes. And by this I mean *completely different* – using combinatorics, algebra, topology, or even analysis with its (multivariate) differential and integral calculus. This is what these notes are about. We will construct the "hole-detecting" *homology* and *cohomology groups* (which will actually be vector spaces over a given ground field in our case) using several entirely different methods, but arriving at the same result in the end. Of course, this is always when things get really interesting since it shows deep relations between different fields of mathematics that seemed to be unrelated before.

Let us quickly explain the intuition behind the algebraic resp. topological and the analytic idea behind homology and cohomology, considering the example $\mathbb{R}^2\backslash\{0\}$ of the punctured plane:

- In *algebra* (resp. *topology*) we consider again the unit circle, just as for the fundamental group. This time however we do not care about its contractibility, but note that it is – roughly speaking – a "closed one-dimensional cycle" (i. e. a loop, as we said above) that is not the boundary of a two-dimensional compact region. In $\mathbb{R}^2$ (without the hole) it would be the boundary of the two-dimensional unit disk, but in $\mathbb{R}^2\backslash\{0\}$ this does not work, showing the existence of the hole. The idea to construct algebraic homology groups is thus to consider "closed cycles modulo boundaries".
- In *analysis*, let us consider the question whether a given smooth (i. e. infinitely differentiable) vector field $f\colon \mathbb{R}^2\backslash\{0\} \to \mathbb{R}^2$ with coordinate functions f_1 and f_2 is the (total) differential of a scalar function $F\colon \mathbb{R}^2\backslash\{0\} \to \mathbb{R}$, i. e. whether there is such a function F with $\partial_1 F = f_1$ and $\partial_2 F = f_2$. Of course, the commutativity of partial derivatives gives the necessary condition $\partial_2 f_1 = \partial_2\partial_1 F = \partial_1\partial_2 F = \partial_1 f_2$ (called the *integrability condition*), but is this also sufficient?

It turns out that on $\mathbb{R}^2$ this would be true, but on $\mathbb{R}^2\backslash\{0\}$ it is not due to the hole in the domain of definition: The "derivative of the angle in polar coordinates", i. e.

$$f(x) = \left(\arctan\frac{x_2}{x_1}\right)' = \left(\frac{-x_2}{x_1^2+x_2^2} \quad \frac{x_1}{x_1^2+x_2^2}\right),$$

would require F to be the angle itself up to an additive constant (which is impossible globally as running around the origin once would pick up another additive constant 2π), although it satisfies the integrability condition since it is locally a derivative. Hence, the analytic idea to detect holes is to consider functions satisfying an integrability condition (i. e. functions that locally look as if they are derivatives) modulo functions that actually are derivatives globally.

To study these approaches and their applications, these notes are divided into four parts:

The first part up to Chapter 5 is purely combinatorial resp. algebraic in nature. In this part we will only consider topological spaces that can be given by combinatorial data in a certain way – similarly to graphs, which can be thought of as one-dimensional topological spaces obtained by gluing finitely many edges at their ends in a specified way. The resulting homology theory is called **simplicial homology**; we will give its construction and study its properties in detail. It is probably the easiest homology theory and thus gives a good introduction to the general ideas and techniques in algebraic topology. Moreover, it has the advantage that everything eventually reduces to linear algebra for finite-dimensional vector spaces, and is thus very accessible for explicit computations.

In the second part from Chapter 6 to 9 we will study **singular homology**, the most general homology theory that is applicable to arbitrary topological spaces. Its construction is very similar to that of simplicial homology and thus mostly algebraic, and in fact with our preparation from the first part many results will just carry over without much effort. We will see that singular homology agrees with simplicial homology when both are defined, and discuss several applications.

We then turn to the analytic setting in the third part. Of course, general topological spaces will not suffice here since they do not have enough structure to define derivatives. From Chapter 10 to 12 we will therefore digress from homology theory and give a short introduction to **smooth manifolds**, which roughly speaking are topological spaces that locally look like $\mathbb{R}^n$. They will allow us, for example, to define tangent spaces at points, leading to the general concept of vector bundles that describe vector spaces varying with a point on a manifold.

In the last part starting in Chapter 13 we will then follow the analytic idea presented above to construct **de Rham cohomology**, the cohomology theory for smooth manifolds. While it shares many of the formal properties of singular cohomology (the dual of singular homology), the proofs of these statements are now analytic, of course, and thus entirely different. Nevertheless we will eventually see that de Rham cohomology agrees with singular cohomology in the cases when both are defined. At the end, we will then use our combined results to prove Poincaré duality – a non-trivial relation among the cohomology groups of oriented manifolds – and to show that cohomology has a product structure that can be described as an "intersection product" given geometrically by the intersection of submanifolds.

As you can see, these notes cover a substantial amount of material – but as usual in topology most statements should be very intuitive. In fact, although singular homology works for arbitrary topological spaces, all topological spaces that we will ever consider in these notes will be very nice in the sense that they can be glued from subsets of $\mathbb{R}^n$ with the standard topology. In other words, our topological spaces will be interesting because they have a non-trivial global structure coming from cutting and gluing processes, and not because they carry a wild topology locally. As a consequence, all our pictures of topological spaces will actually visualize correctly what is going on, and many results that we will prove are probably already plausible from the pictures.

We have mentioned already that algebraic topology has many relations to a wide variety of mathematical areas – which is after all what makes it so interesting! In order to keep these notes accessible we will therefore usually refrain from discussing every topic in full breadth and generality, include many examples, and keep the prerequisites to a minimum. Let us quickly list them.

Remark 0.1 (Prerequisites)**.** In these notes we will roughly assume the knowledge of a first-year mathematics class on algebra and analysis, together with an introductory course in topology. More precisely, we will assume familiarity with the following topics:

(a) *Algebra:* permutation groups, the finite fields $\mathbb{Z}/p\mathbb{Z}$, vector spaces and linear maps, determinants, dual vector spaces.

(b) *Analysis:* multivariate differentiation, implicit functions, multivariate integration in $\mathbb{R}^n$. As for integration, Riemann and Lebesgue integrals will be equally fine since all our functions will be nice enough so that the difference does not matter.

(c) *Topology:* topological spaces and their basic properties, quotient spaces, homotopies and homotopy equivalences. Fundamental groups might help since they share the general idea with homology groups of assigning algebraic objects to topological spaces, but they will not actually be used.

Many connections to other fields of mathematics will be mentioned throughout the notes, but prior knowledge of them will not be required: Simplicial complexes are basically just a straightforward generalization of *graph theory*. Relating mathematical objects of different types such as assigning vector spaces to topological spaces is one of the main ideas of *category theory*. The chain complexes and exact sequences needed to construct homology are a topic from *commutative* resp. *homological algebra* – but clearly motivated by topology, so if you have seen them already then our topological point of view will probably make them more intuitive. A byproduct of our analytic study of cohomology will be the multivariate integral theorems from *vector analysis* such as the divergence theorem of Gauss and the curl theorem of Stokes. The study of topological properties of manifolds is an important part of *algebraic geometry*. Finally, on the applied mathematics side, *topological data analysis* uses algebraic topology in the form of persistent homology as its backbone.

Notation 0.2. Throughout these notes we will adopt the following common notations and conventions:

- The set $\mathbb{N}$ of natural numbers contains 0.
- For two sets S and T the inclusion relation $T \subset S$ allows the case $T = S$; we write $T \subsetneq S$ if equality is not allowed.
- K will always be a fixed ground field. Our usual choice will be $K = \mathbb{R}$, and not much will be lost if you just always assume that. Sometimes we will also consider the finite field $\mathbb{Z}/p\mathbb{Z}$ for a prime number p, which we denote by $\mathbb{Z}_p$.
- For a finite index set I with cardinality $n \in \mathbb{N}$, we denote by
$$K^I := \{x = (x_i)_{i\in I} : x_i \in K \text{ for all } i \in I\}$$
the n-dimensional K-vector space with coordinates labeled by I. Its standard basis is $(e_i)_{i\in I}$, where $e_i \in K^I$ for $i \in I$ is the standard unit vector whose i-th component is 1, and all others are 0. If $J \subset I$ is another index set, we regard K^J as a vector subspace of K^I by setting all coordinates in $J\backslash I$ to 0.

  Of course, the most common case will be $K^n := K^{\{1,\dots,n\}}$ for $n \in \mathbb{N}$. For $n = 0$, the zero vector space K^0 is often just written as 0 instead of $\{0\}$.
- More generally, for any (not necessarily finite) set B we can construct the vector space of all tuples $(x_b)_{b\in B}$ with $x_b \in K$ for all $b \in B$ such that only finitely many x_b are non-zero. Usually, for a given $b \in B$ the tuple with $x_b = 1$ and all other coordinates being 0 is then just written as b, so that every element of the vector space can be thought of as a finite linear combination $\sum_{b\in B} x_b \cdot b$ (i. e. with almost all $x_b = 0$). In particular, this means that B itself is a basis of this vector space. We will refer to it as *the vector space with basis B*.
- For a vector space V, its dual space will be written as $V^\vee$ (writing it as V^*, which is common in many areas of mathematics, would be confusing later on since the star notation will already have a different meaning as e. g. in Construction 5.6).

- For $n \in \mathbb{N}$, the vector spaces $\mathbb{R}^n$ and any of their subsets will always be equipped with the standard topology coming from the usual (Euclidean) norm $\|\cdot\|_2$, which we just write as $\|\cdot\|$. As special subsets, we denote by $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ the unit n-sphere in $\mathbb{R}^{n+1}$, and by $D^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ the unit n-disk (or n-ball) in $\mathbb{R}^n$. The open ball with radius $r \in \mathbb{R}_{>0}$ and center point $a \in \mathbb{R}^n$ will be written as $U_r(a)$.