Algebraic Geometry – Problem Set 9

due Thursday, January 13

(1) Prove the projective Jacobi criterion:
Let $X \subset \mathbb{P}^n$ be a projective variety with ideal $I(X) = \langle f_1, \dots, f_r \rangle$, and let $a \in X$. Then X is smooth at a if and only if the rank of the $r \times (n+1)$ Jacobi matrix $\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j}$ is at least $n - \operatorname{codim}_X\{a\}$.

Hint: Show and use that $\sum_{i=0}^n x_i \cdot \frac{\partial f}{\partial x_i} = df$ for every homogeneous polynomial $f \in K[x_0, \dots, x_n]$ of degree d.

(2) For $k \in \mathbb{N}_{>0}$ let X_k be the complex singular affine curve $X_k := V(x_2^2 - x_1^{2k+1}) \subset \mathbb{A}^2_{\mathbb{C}}$, and denote by $\widetilde{X}_k \subset \widetilde{\mathbb{A}^2}$ the blow-ups of X_k and $\mathbb{A}^2$ at the origin, respectively.

(a) Use suitable coordinates on $\widetilde{\mathbb{A}^2}$ to determine all k for which $\widetilde{X}_k$ is smooth.

(b) Show that X_k is not isomorphic to X_l if $k \neq l$.

Hint: Follow the idea of Example 10.16 in the notes (which we skipped in class).

(3) Let $n \geq 2$. Prove:

(a) Every smooth hypersurface in $\mathbb{P}^n$ is irreducible.

(b) A general hypersurface in $\mathbb{P}^n_{\mathbb{C}}$ is smooth (and thus by (a) irreducible). More precisely, for a given $d \in \mathbb{N}_{>0}$ the vector space $\mathbb{C}[x_0, \dots, x_n]_d$ has dimension $\binom{n+d}{n}$, and so the space of all homogeneous degree-d polynomials in $x_0, \dots, x_n$ modulo scalars can be identified with the projective space $\mathbb{P}_{\mathbb{C}}^{\binom{n+d}{n}-1}$. Show that the subset of this projective space of all (classes of) polynomials f such that f is irreducible and $V_p(f)$ is smooth is dense and open.

(4) Assume that the characteristic of K is not equal to 2, and let $f \in K[x_0, x_1, x_2]$ be a homogeneous polynomial whose partial derivatives $\frac{\partial f}{\partial x_i}$ for $i = 0, 1, 2$ do not vanish simultaneously at any point of $X = V_p(f) \subset \mathbb{P}^2$. Then the image of the morphism

$$F\colon X \to \mathbb{P}^2,\ a \mapsto \left(\frac{\partial f}{\partial x_0}(a) : \frac{\partial f}{\partial x_1}(a) : \frac{\partial f}{\partial x_2}(a)\right)$$

is called the *dual curve* to X.

(a) Find a geometric description of F. What does it mean geometrically if $F(a) = F(b)$ for two distinct points $a, b \in X$?

(b) If X is a conic (i. e. an irreducible curve of degree 2), prove that its dual $F(X)$ is also a conic.

(c) For any five lines in $\mathbb{P}^2$ in general position (what does this mean?) show that there is a unique conic in $\mathbb{P}^2$ that is tangent to all of them.