Algebraic Geometry – Problem Set 8

due Thursday, January 6

(1) Let $\widetilde{\mathbb{A}^3}$ be the blow-up of $\mathbb{A}^3$ at the line $V(x_1,x_2) \cong \mathbb{A}^1$. Show that its exceptional set is isomorphic to $\mathbb{A}^1 \times \mathbb{P}^1$. When do the strict transforms of two lines in $\mathbb{A}^3$ through $V(x_1,x_2)$ intersect in the blow-up? What is therefore the geometric meaning of the points in the exceptional set (analogously to the blow-up of a point, in which case the points of the exceptional set correspond to the directions through the blown-up point)?

(2) Show that any irreducible quadric hypersurface in $\mathbb{P}^n$ over a field of characteristic not equal to 2 is birational, but in general not isomorphic to the projective space $\mathbb{P}^{n-1}$.

(3) Let $X \subset \mathbb{A}^n$ be an affine variety, and let $Y_1,Y_2 \subsetneq X$ be irreducible, closed subsets, no-one contained in the other. Moreover, let $\tilde{X}$ be the blow-up of X at the ideal $I(Y_1)+I(Y_2)$.
Show that the strict transforms of Y_1 and Y_2 in $\tilde{X}$ are disjoint.

(4) Let $J \trianglelefteq K[x_1,\dots,x_n]$ be an ideal, and assume that the corresponding affine variety $X = V(J) \subset \mathbb{A}^n$ contains the origin. Consider the blow-up $\tilde{X} \subset \widetilde{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ at $x_1,\dots,x_n$, and denote the homogeneous coordinates of $\mathbb{P}^{n-1}$ by $y_1,\dots,y_n$.

(a) We know already that $\widetilde{\mathbb{A}^n}$ can be covered by affine spaces, with one coordinate patch being

$$\mathbb{A}^n \to \widetilde{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{P}^{n-1},$$
$$(x_1,y_2,\dots,y_n) \mapsto ((x_1,x_1y_2,\dots,x_1y_n),(1:y_2:\dots:y_n)).$$

Prove that on this coordinate patch the blow-up $\tilde{X}$ is given as the zero locus of the polynomials

$$\frac{f(x_1,x_1y_2,\dots,x_1y_n)}{x_1^{\min\deg f}}$$

for all non-zero $f \in J$, where $\min\deg f$ denotes the smallest degree of a monomial in f.

(b) Show that the exceptional set of the blow-up $\tilde{X}$ is

$$V_{\mathrm{p}}(f^{\mathrm{in}}(y) : f \in J) \quad \subset \mathbb{P}^{n-1} \cong \{0\} \times \mathbb{P}^{n-1},$$

where f^{in} is the *initial term* of f, i. e. the sum of all monomials in f of smallest degree. Consequently, the tangent cone of X at the origin is

$$C_0X = V_{\mathrm{a}}(f^{\mathrm{in}} : f \in J) \quad \subset \mathbb{A}^n.$$