Andreas Gathmann and Jonas Frank Winter Semester 2021/22

Algebraic Geometry – Problem Set 7

due Thursday, December 16

(1) Let $X \subset \mathbb{P}^3$ be the degree-3 Veronese embedding of $\mathbb{P}^1$, i. e. the image of the morphism

$$\mathbb{P}^1 \to \mathbb{P}^3,\ (x_0 : x_1) \mapsto (y_0 : y_1 : y_2 : y_3) = (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3).$$

Moreover, let $a = (0:0:1:0) \in \mathbb{P}^3$ and $L = V(y_2) \subset \mathbb{P}^3$, and let f be the projection from a to L.

(a) Determine an equation of the curve $f(X)$ in $L \cong \mathbb{P}^2$.

(b) Is $f\colon X \to f(X)$ an isomorphism onto its image?

(2) (a) For any $n, d \in \mathbb{N}_{>0}$, find explicit equations describing the image of the degree-d Veronese embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$.

(b) Prove that every projective variety is isomorphic to the zero locus of *quadratic* polynomials in a projective space.

(3) We denote the Plücker coordinates of the Grassmannian $G(2,4)$ in $\mathbb{P}^5$ by $x_{i,j}$ for $1 \le i < j \le 4$.

(a) Show that $G(2,4) = V(x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3})$.

(b) Let $L \subset \mathbb{P}^3$ be an arbitrary line. Show that the set of lines in $\mathbb{P}^3$ that intersect L, considered as a subset of $G(2,4) \subset \mathbb{P}^5$, is the zero locus of a homogeneous linear polynomial.

How many lines in $\mathbb{P}^3$ would you expect to intersect four general given lines?

(4) Show that the following sets are projective varieties:

(a) the *incidence correspondence*

$$\{(L,a) \in G(k,n) \times \mathbb{P}^{n-1} : L \subset \mathbb{P}^{n-1} \text{ a } (k-1)\text{-dimensional linear subspace and } a \in L\};$$

(b) the *join* of two disjoint varieties $X, Y \subset \mathbb{P}^n$, i. e. the union of all lines in $\mathbb{P}^n$ intersecting both X and Y.