Andreas Gathmann and Jonas Frank Winter Semester 2021/22

# *Algebraic Geometry – Problem Set 6*

*due Thursday, December 9*

(1) Let $m,n \in \mathbb{N}_{>0}$. Prove:

(a) If $f\colon \mathbb{P}^n \to \mathbb{P}^m$ is a morphism and $X \subset \mathbb{P}^m$ a hypersurface then every irreducible component of $f^{-1}(X)$ has dimension at least $n-1$.

(b) If $n > m$ then every morphism $f\colon \mathbb{P}^n \to \mathbb{P}^m$ is constant.

(c) $\mathbb{P}^n \times \mathbb{P}^m$ is not isomorphic to $\mathbb{P}^{n+m}$.

(2) Let us say that $n+2$ points in $\mathbb{P}^n$ are *in general position* if for any $n+1$ of them their representatives in $K^{n+1}$ are linearly independent.

Now let $a_1,\dots,a_{n+2}$ and $b_1,\dots,b_{n+2}$ be two sets of points in $\mathbb{P}^n$ in general position. Show that there is an isomorphism $f\colon \mathbb{P}^n \to \mathbb{P}^n$ with $f(a_i) = b_i$ for all $i = 1,\dots,n+2$.

(3) Show by example that the homogeneous coordinate ring of a projective variety is *not* invariant under isomorphisms, i.e. that there are isomorphic projective varieties $X,Y$ such that the rings $S(X)$ and $S(Y)$ are not isomorphic.

(4) A *conic* over a field of characteristic not equal to 2 is an irreducible curve in $\mathbb{P}^2$ of degree 2.

(a) Using the coefficients of quadratic polynomials, show that the set of all conics can be identified with an open subset $U$ of $\mathbb{P}^5$. (One says that $U$ is a *moduli space* for conics.)

(b) Given a point $a \in \mathbb{P}^2$, show that the subset of $U$ consisting of all conics passing through $a$ is the zero locus of a linear equation in the homogeneous coordinates of $U \subset \mathbb{P}^5$.

(c) Given 5 points in $\mathbb{P}^2$, no three of which lie on a line, show that there is a unique conic passing through all these points.