Andreas Gathmann and Jonas Frank Winter Semester 2021/22

Algebraic Geometry – Problem Set 5

due Thursday, December 2

(1) (a) Compute explicit generators for the ideal $I(a) \trianglelefteq K[x_0,\dots,x_n]$ of an arbitrary point $a \in \mathbb{P}^n$.

(b) Let $X = V(x_1^2 - x_2^2 - 1, x_3 - x_1) \subset \mathbb{A}^3_{\mathbb{C}}$. What are the points at infinity of the projective closure $\overline{X} \subset \mathbb{P}^3_{\mathbb{C}}$, i. e. the points in $\overline{X} \backslash X$?

(2) A *line* in $\mathbb{P}^n$ is a linear subspace of dimension 1.

Let $L_1, L_2 \subset \mathbb{P}^3$ be two disjoint lines, and let $a \in \mathbb{P}^3 \backslash (L_1 \cup L_2)$. Show that there is a unique line $L \subset \mathbb{P}^3$ through a that intersects both L_1 and L_2.

Is the corresponding statement for lines and points in $\mathbb{A}^3$ true as well?

(3) (a) Prove that a graded ring R is an integral domain if and only if for all *homogeneous* elements $f, g \in R$ with $fg = 0$ we have $f = 0$ or $g = 0$.

(b) Show that a projective variety X is irreducible if and only if its homogeneous coordinate ring $S(X)$ is an integral domain.

(4) In this exercise we want to show that an intersection of projective varieties is never empty unless one would expect it to be empty for dimensional reasons — so e. g. the phenomenon of parallel non-intersecting lines in the plane does not occur in projective space.

(a) Let $X, Y \subset \mathbb{A}^n$ be pure-dimensional affine varieties. Show that every irreducible component of $X \cap Y$ has dimension at least $\dim X + \dim Y - n$.

(b) Now let $X \subset \mathbb{P}^n$ be a projective variety. Prove that the dimension of the cone $C(X) \subset \mathbb{A}^{n+1}$ is $\dim X + 1$.

(c) Let $X, Y \subset \mathbb{P}^n$ be projective varieties with $\dim X + \dim Y \geq n$. Show that $X \cap Y \neq \emptyset$.

(Hint: We have assumed from commutative algebra that $\dim(X \times Y) = \dim X + \dim Y$ for any two affine varieties X and Y.)