Andreas Gathmann and Jonas Frank Winter Semester 2021/22

Algebraic Geometry – Problem Set 4

due Thursday, November 25

As in class let $\mathbb{P}^1$ be the prevariety obtained by gluing two copies of the affine line $\mathbb{A}^1$ along the isomorphism $\mathbb{A}^1\backslash\{0\} \to \mathbb{A}^1\backslash\{0\}$, $x \mapsto \frac{1}{x}$. By the inclusion of one of the copies we consider $\mathbb{A}^1$ as an open subprevariety of $\mathbb{P}^1$.

(1) Which of the following ringed spaces are isomorphic over $\mathbb{C}$?

(a) $\mathbb{A}^1$
(b) $V(x_1^2+x_2^2) \subset \mathbb{A}^2$
(c) $V(x_2-x_1^2, x_3-x_1^3)\backslash\{0\} \subset \mathbb{A}^3$
(d) $V(x_1x_2) \subset \mathbb{A}^2$
(e) $\mathbb{A}^1\backslash\{1\}$

(2) Let $f\colon X \to Y$ be a morphism of affine varieties and $f^*\colon A(Y) \to A(X)$ the corresponding homomorphism of the coordinate rings. Are the following statements true or false?

(a) f is surjective if and only if f^* is injective.

(b) f is injective if and only if f^* is surjective.

(c) If $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism then f is affine linear, i. e. of the form $f(x) = ax+b$ for some $a,b \in K$.

(d) If $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ is an isomorphism then f is affine linear, i. e. it is of the form $f(x) = Ax+b$ for some $A \in \mathrm{Mat}(2\times 2, K)$ and $b \in K^2$.

(3) Prove the following statements:

(a) Every morphism $\mathbb{A}^1\backslash\{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^1 \to \mathbb{P}^1$.

(b) Not every morphism $\mathbb{A}^2\backslash\{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^2 \to \mathbb{P}^1$.

(c) Every morphism $\mathbb{P}^1 \to \mathbb{A}^1$ is constant.

(4) If X and Y are affine varieties we have seen that there is a bijection

$$\{\text{morphisms } X \to Y\} \overset{1:1}{\longleftrightarrow} \{K\text{-algebra homomorphisms } \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)\}$$
$$f \longmapsto f^*.$$

Does this statement still hold

(a) if X is an arbitrary prevariety (but Y is still affine);

(b) if Y is an arbitrary prevariety (but X is still affine)?