Andreas Gathmann and Jonas Frank Winter Semester 2021/22

Algebraic Geometry – Problem Set 3

due Thursday, November 18

(1) Let $\varphi, \psi \in \mathscr{F}(U)$ be two sections of a sheaf $\mathscr{F}$ on an open subset U of a topological space X. Show:

(a) If φ and ψ agree in all stalks, i. e. $\overline{(U,\varphi)} = \overline{(U,\psi)} \in \mathscr{F}_a$ for all $a \in U$, then $\varphi = \psi$.

(b) If $\mathscr{F} = \mathscr{O}_X$ is the sheaf of regular functions on an irreducible affine variety X then we can already conclude that $\varphi = \psi$ if we only know that they agree in *one* stalk $\mathscr{F}_a$ for $a \in U$.

(c) For a general sheaf $\mathscr{F}$ on a topological space X the statement of (b) is false.

(2) Let a be any point on the real line $\mathbb{R}$. For which of the following sheaves $\mathscr{F}$ on $\mathbb{R}$ (with the standard topology) is the stalk $\mathscr{F}_a$ actually a local ring in the algebraic sense (i. e. it has exactly one maximal ideal)?

(a) $\mathscr{F}$ is the sheaf of continuous functions;

(b) $\mathscr{F}$ is the sheaf of locally polynomial functions.

(3) Let Y be a non-empty irreducible subvariety of an affine variety X, and set $U = X \backslash Y$.

(a) Assume that $A(X)$ is a unique factorization domain. Show that $\mathscr{O}_X(U) = A(X)$ if and only if $\operatorname{codim} Y \geq 2$.

(b) Show by example that the equivalence of (a) is in general false if $A(X)$ is not assumed to be a unique factorization domain.

(4) Let $\mathscr{F}$ be a sheaf on a topological space X, and let Y be a non-empty irreducible closed subset of X. We define the *stalk of $\mathscr{F}$ at Y* to be

$$\mathscr{F}_Y := \{(U,\varphi) : U \text{ is an open subset of } X \text{ with } U \cap Y \neq \emptyset, \text{ and } \varphi \in \mathscr{F}(U)\} \,/ \sim$$

where $(U,\varphi) \sim (U',\varphi')$ if and only if there is an open set $V \subset U \cap U'$ with $V \cap Y \neq \emptyset$ and $\varphi|_V = \varphi'|_V$. It therefore describes functions in an arbitrarily small neighborhood of an arbitrary dense open subset of Y.

If Y is a non-empty irreducible subvariety of an affine variety X, prove that the stalk $\mathscr{O}_{X,Y}$ of $\mathscr{O}_X$ at Y is a K-algebra isomorphic to the localization $A(X)_{I(Y)}$ (hence giving a geometric meaning to this algebraic localization).