Algebraic Geometry – Problem Set 2

due Thursday, November 11

(1) Find the irreducible components of the affine variety $V(x_1 - x_2x_3, x_1x_3 - x_2^2) \subset \mathbb{A}^3_{\mathbb{C}}$.

(2) We set

$$X := \{A \in \mathrm{Mat}(2\times 2,\mathbb{C}) : A \text{ is nilpotent}\} \quad \subset \mathrm{Mat}(2\times 2,\mathbb{C}) = \mathbb{A}^4_{\mathbb{C}}$$
$$\text{and} \quad Y := \{A \in \mathrm{Mat}(2\times 3,\mathbb{C}) : \mathrm{rk}\, A \le 1\} \quad \subset \mathrm{Mat}(2\times 3,\mathbb{C}) = \mathbb{A}^6_{\mathbb{C}}.$$

Show that X and Y are irreducible affine varieties, and compute their dimensions.

(3) Let U be a non-empty subset of an affine variety X. Prove:

(a) U is irreducible if and only if its closure $\overline{U}$ is irreducible.

(b) If U is open then $\dim U = \dim \overline{U}$.

Which of these statements still hold if X is a general topological space?

(4) Recall that for two ideals J_1 and J_2 in a ring R the *ideal quotient* is defined by

$$J_1 : J_2 = \{f \in R : fJ_2 \subset J_1\}.$$

Show that ideal quotients correspond to differences of varieties in the following sense: If X is an affine variety and ...

(a) Y_1 and Y_2 are subvarieties of X then $I(\overline{Y_1 \backslash Y_2}) = I(Y_1) : I(Y_2)$ in $A(X)$;

(b) J_1 and J_2 are radical ideals in $A(X)$ then $\overline{V(J_1)\backslash V(J_2)} = V(J_1 : J_2)$.