Andreas Gathmann and Jonas Frank Winter Semester 2021/22

Algebraic Geometry – Problem Set 12

due Thursday, February 3

(1) (a) Find $d \in \mathbb{Z}$ and morphisms f and g such that the sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1} \xrightarrow{f} \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \xrightarrow{g} \mathscr{O}_{\mathbb{P}^1}(d) \longrightarrow 0$$

is exact on $\mathbb{P}^1$.

(b) Let $X \subset \mathbb{P}^n$ be a projective hypersurface of degree d. Show that the ideal sheaf of X in $\mathbb{P}^n$ is given by $\mathscr{I}_{X/\mathbb{P}^n} \cong \mathscr{O}_{\mathbb{P}^n}(-d)$.

(2) Let $n \in \mathbb{N}_{>0}$ and $d, e \in \mathbb{Z}$. Prove:

(a) $\mathscr{O}_{\mathbb{P}^n}(d) \otimes \mathscr{O}_{\mathbb{P}^n}(e) \cong \mathscr{O}_{\mathbb{P}^n}(d+e)$;

(b) $\mathscr{O}_{\mathbb{P}^n}(d)^\vee \cong \mathscr{O}_{\mathbb{P}^n}(-d)$.

(3) Let $f \colon \mathscr{F} \to \mathscr{G}$ be an injective morphism of sheaves on a scheme X. Prove that then the image presheaf $\operatorname{Im}' f$ is already a sheaf.

Conclude from this that if $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3$ is an exact sequence of sheaves on X then the sequence of sections $0 \to \mathscr{F}_1(U) \to \mathscr{F}_2(U) \to \mathscr{F}_3(U)$ is also exact for all open subsets $U \subset X$.

(4) Let $\mathscr{F}'$ be a presheaf on a scheme X, and denote by $\theta \colon \mathscr{F}' \to \mathscr{F}$ its sheafification.

Prove that any morphism $f' \colon \mathscr{F}' \to \mathscr{G}$ to a sheaf $\mathscr{G}$ factors uniquely through θ, i. e. that there is a unique morphism $f \colon \mathscr{F} \to \mathscr{G}$ with $f' = f \circ \theta$.