Andreas Gathmann and Jonas Frank Winter Semester 2021/22

Algebraic Geometry – Problem Set 11

due Thursday, January 27

(1) Show that for a scheme X the following are equivalent:

(a) X is reduced, i. e. for every open subset $U \subset X$ the ring $\mathscr{O}_X(U)$ has no nilpotent elements.

(b) There is an open cover of X by affine schemes $U_i = \operatorname{Spec} R_i$ such that every ring $\mathscr{O}_X(U_i) = R_i$ has no nilpotent elements.

(c) For every point $P \in X$ the local ring $\mathscr{O}_{X,P}$ has no nilpotent elements.

(2) For $n \in \mathbb{N}_{>0}$, an n-fold point over an algebraically closed field K is a scheme over K of the form $\operatorname{Spec} R$ that contains only one point, and such that R is a K-algebra of vector space dimension n over K.

(a) Show that every double point over K is isomorphic to $\operatorname{Spec} K[x]/\langle x^2 \rangle$.

(b) Find two non-isomorphic triple points over K. Can you describe them geometrically?

(3) Any morphism $f\colon \mathscr{F} \to \mathscr{G}$ of sheaves of modules on a scheme X determines induced $\mathscr{O}_{X,P}$-module homomorphisms $f_P\colon \mathscr{F}_P \to \mathscr{G}_P$ on the stalks for all $P \in X$, by mapping a germ $\overline{(U,\varphi)} \in \mathscr{F}_P$ for an open neighborhood U of P and $\varphi \in \mathscr{F}(U)$ to $\overline{(U, f_U(\varphi))} \in \mathscr{G}_P$.

Show that f is an isomorphism if and only if all f_P are isomorphisms.

(4) Let $\mathscr{F}$ and $\mathscr{G}$ be two sheaves of modules on a scheme X, and let $f\colon \mathscr{F} \to \mathscr{G}$ be a morphism. Show:

(a) The *kernel presheaf* $\operatorname{Ker} f$, defined by $(\operatorname{Ker} f)(U) = \operatorname{Ker}(f_U\colon \mathscr{F}(U) \to \mathscr{G}(U))$, is actually a sheaf.

(b) The *image presheaf* $\operatorname{Im}' f$, defined by $(\operatorname{Im}' f)(U) = \operatorname{Im}(f_U\colon \mathscr{F}(U) \to \mathscr{G}(U))$, is in general not a sheaf.

(c) The *tensor presheaf* $\mathscr{F} \otimes' \mathscr{G}$, defined by $(\mathscr{F} \otimes' \mathscr{G})(U) = \mathscr{F}(U) \otimes \mathscr{G}(U)$, is in general not a sheaf.