Andreas Gathmann and Jonas Frank Winter Semester 2021/22

Algebraic Geometry – Problem Set 10

due Thursday, January 20

(1) Find an example of the following, or prove that it does not exist:

(a) an irreducible affine scheme $\operatorname{Spec} R$ such that R is not an integral domain;

(b) a point of $\operatorname{Spec} \mathbb{R}[x_1,x_2]/\langle x_1^2+x_2^2+1\rangle$ with residue field $\mathbb{R}$;

(c) two affine schemes $\operatorname{Spec} R$ and $\operatorname{Spec} S$ with $R \leq S$ and $\dim \operatorname{Spec} R > \dim \operatorname{Spec} S$;

(d) an affine scheme of dimension 1 with exactly two points.

(2) (a) Let $R = A(X)$ be the coordinate ring of an affine variety X over an algebraically closed field. Show that the set of all closed points is dense in $\operatorname{Spec} R$ (which means by definition that every non-empty open subset of $\operatorname{Spec} R$ contains a closed point).

(b) In contrast to (a) however, show by example that on a general affine scheme the set of all closed points need not be dense.

(3) Let R be a ring. Prove that the affine scheme $\operatorname{Spec} R$ is disconnected if and only if $R \cong S \times T$ for two non-zero rings S and T.

(4) Let R be a ring, and let $f \in R$. Show that every regular function on the distinguished open subset $D(f)$ of the affine scheme $\operatorname{Spec} R$ is of the form $\frac{g}{f^n}$ for some $g \in R$ and $n \in \mathbb{N}$.
(Hint: Adapt the proof from class of the corresponding statement for affine varieties.)