Andreas Gathmann and Jonas Frank Winter Semester 2021/22

Algebraic Geometry – Problem Set 1

due Thursday, November 4

(1) Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes.

(a) Compute generators for the ideal $I(X)$.

(b) Show that $I(X)$ cannot be generated by fewer than three elements.

(2) Let $Y \subset \mathbb{A}^n$ be an affine variety, and denote by $\pi\colon K[x_1,\dots,x_n] \to K[x_1,\dots,x_n]/I(Y) = A(Y)$ the quotient map.

(a) Show that $V_Y(J) = V(\pi^{-1}(J))$ for every ideal J in $A(Y)$.

(b) Show that $\pi^{-1}(I_Y(X)) = I(X)$ for every subvariety X of Y.

(c) Use (a) and (b) to deduce the (interesting part of the) relative Nullstellensatz $I_Y(V_Y(J)) = \sqrt{J}$ for every ideal $J \trianglelefteq A(Y)$ from the corresponding absolute statement $I(V(J)) = \sqrt{J}$ for every ideal $J \trianglelefteq K[x_1,\dots,x_n]$. In particular, conclude that there is an inclusion-reversing bijection between affine subvarieties of Y and radical ideals in $A(Y)$.

(3) Prove that every affine variety $X \subset \mathbb{A}^n$ consisting of only finitely many points can be written as the zero locus of n polynomials.

(Hint: Use interpolation. It is useful to assume first that all points in X have different x_1-coordinates.)

(4) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be irreducible affine varieties. Show that their product variety $X \times Y \subset \mathbb{A}^{n+m}$ is irreducible as well.

Please upload your solutions until the due date (at any time) in the OLAT course as a single PDF file.