9. Birational Maps and Blowing Up

In the course of this class we have already seen many examples of varieties that are "almost the same" in the sense that they contain isomorphic dense open subsets (although the varieties are not isomorphic themselves). Let us quickly recall some of them.

Example 9.1 (Irreducible varieties with isomorphic non-empty open subsets)**.**

(a) The affine space $\mathbb{A}^n$ and the projective space $\mathbb{P}^n$ have the common open subset $\mathbb{A}^n$ by Proposition 7.2. Consequently, $\mathbb{P}^m \times \mathbb{P}^n$ and $\mathbb{P}^{m+n}$ have the common open subset $\mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$ — but they are not isomorphic by Exercise 7.7 (c).

(b) Similarly, the affine space $\mathbb{A}^{k(n-k)}$ and the Grassmannian $G(k,n)$ have the common open subset $\mathbb{A}^{k(n-k)}$ by Construction 8.18.

(c) The affine line $\mathbb{A}^1$ and the cubic curve $X = V(x_1^2 - x_2^3) \subset \mathbb{A}^2$ of Example 4.9 have the isomorphic open subsets $\mathbb{A}^1 \backslash \{0\}$ resp. $X \backslash \{0\}$ — in fact, the morphism f given there is an isomorphism after removing the origin from both the source and the target curve.

We now want to study this situation in more detail and present a very general construction — the so-called blow-ups — that gives rise to many examples of this type. But first of all we have to set up some notation to deal with morphisms that are defined on dense open subsets. For simplicity, we will do this only for the case of irreducible varieties, in which every non-empty open subset is automatically dense by Remark 2.16.

Definition 9.2 (Rational maps)**.** Let X and Y be irreducible varieties. A **rational map** f from X to Y, written $f\colon X \dashrightarrow Y$, is a morphism $f\colon U \to Y$ (denoted by the same letter) from a non-empty open subset $U \subset X$ to Y. We say that two such rational maps $f_1, f_2\colon X \dashrightarrow Y$ defined on U_1 resp. U_2 are the same if $f_1 = f_2$ on a non-empty open subset of $U_1 \cap U_2$.

Remark 9.3.

(a) Strictly speaking, Definition 9.2 means that a rational map $f\colon X \dashrightarrow Y$ is an equivalence class of morphisms from non-empty open subsets of X to Y. Note that the given relation is in fact an equivalence relation: Reflexivity and symmetry are obvious, and if $f_1\colon U_1 \to Y$ agrees with $f_2\colon U_2 \to Y$ on a non-empty open subset $U_{1,2}$ and f_2 with $f_3\colon U_3 \to Y$ on a non-empty open subset $U_{2,3}$ then f_1 and f_3 agree on $U_{1,2} \cap U_{2,3}$, which is again non-empty by Remark 2.16 (a) since X is irreducible.

For the sake of readability it is customary however not to indicate these equivalence classes in the notation and to denote the rational map $f\colon X \dashrightarrow Y$ and the morphism $f\colon U \to Y$ by the same letter.

(b) If two rational maps $f_1, f_2\colon X \dashrightarrow Y$ defined on $U_1 \subset X$ resp. $U_2 \subset X$ are the same, i. e. if they agree on a non-empty open subset $U \subset U_1 \cap U_2$, note that they must already agree on their full common domain of definition $U_1 \cap U_2$ since this is the closure of U in $U_1 \cap U_2$ and the locus where two morphisms agree is closed by Proposition 5.20 (b).

Example 9.4. The morphism $f\colon \mathbb{A}^1 \backslash \{0\} \to \mathbb{A}^1$, $x \mapsto \frac{1}{x}$ defines a rational map $f\colon \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$. The
14 morphism $\mathbb{A}^1 \backslash \{0,1\} \to \mathbb{A}^1$, $x \mapsto \frac{1}{x}$ represents the same rational map from $\mathbb{A}^1$ to itself.

If we now want to consider "rational maps with an inverse", i. e. rational maps $f\colon X \dashrightarrow Y$ such that there is another rational map $g\colon Y \dashrightarrow X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, we run into problems: If e. g. f is a constant map and g is not defined at the point $f(X)$ then there is no meaningful way to compose it with f. So we need to impose a technical condition first to ensure that compositions are well-defined:

Definition 9.5 (Birational maps)**.** Again let X and Y be irreducible varieties.

(a) A rational map $f\colon X \dashrightarrow Y$ is called **dominant** if its image contains a non-empty open subset U of Y. In this case, if $g\colon Y \dashrightarrow Z$ is another rational map, defined on a non-empty open subset V of Y, we can construct the composition $g \circ f\colon X \dashrightarrow Z$ as a rational map since we have such a composition of ordinary morphisms on the non-empty open subset $f^{-1}(U \cap V)$.

(b) A rational map $f\colon X \dashrightarrow Y$ is called **birational** if it is dominant, and if there is another dominant rational map $g\colon Y \dashrightarrow X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

(c) We say that X and Y are **birational** if there is a birational map $f\colon X \dashrightarrow Y$ between them.

Remark 9.6. Note that two irreducible varieties are birational if and only if they contain isomorphic non-empty open subsets:

"$\Rightarrow$" If $f\colon X \dashrightarrow Y$ is a birational map defined on $U \subset X$, with inverse $g\colon Y \dashrightarrow X$ defined on $V \subset Y$, the open subsets $U \cap f^{-1}(V) \subset X$ and $V \cap g^{-1}(U) \subset Y$ are isomorphic by f and g.

"$\Leftarrow$" If $f\colon U \to V$ is an isomorphism between non-empty open subsets $U \subset X$ and $V \subset Y$, then f is by definition a birational map $f\colon X \dashrightarrow Y$.

In particular, Exercise 5.24 then implies that birational irreducible varieties have the same dimension.

An important case of rational maps is when the target space is just the ground field, i. e. if we consider regular functions on open subsets.

Construction 9.7 (Rational functions and function fields)**.** Let X be an irreducible variety.

A rational map $\varphi\colon X \dashrightarrow \mathbb{A}^1 = K$ is called a **rational function** on X. In other words, a rational function on X is given by a regular function $\varphi \in \mathscr{O}_X(U)$ on some non-empty open subset $U \subset X$, with two such regular functions defining the same rational function if and only if they agree on a non-empty open subset. The set of all rational functions on X will be denoted $K(X)$.

Note that $K(X)$ is a field: For $\varphi_1 \in \mathscr{O}_X(U_1)$ and $\varphi_2 \in \mathscr{O}_X(U_2)$ we can define $\varphi_1 + \varphi_2$ and $\varphi_1\,\varphi_2$ on $U_1 \cap U_2 \neq \emptyset$, the additive inverse $-\varphi_1$ on U_1, and for $\varphi_1 \neq 0$ the multiplicative inverse φ_1^{-1} on $U_1 \backslash V(\varphi_1)$. We call $K(X)$ the **function field** of X.

Remark 9.8.

(a) If $U \subset X$ is a non-empty open subset of an irreducible variety X then $K(U) \cong K(X)$: An isomorphism is given by

$$\begin{aligned} K(U) &\to K(X) \\ \varphi \in \mathscr{O}_U(V) &\mapsto \varphi \in \mathscr{O}_X(V) \end{aligned} \qquad \text{with inverse} \qquad \begin{aligned} K(X) &\to K(U) \\ \varphi \in \mathscr{O}_X(V) &\mapsto \varphi|_{V\cap U} \in \mathscr{O}_U(V \cap U). \end{aligned}$$

In particular, birational irreducible varieties have isomorphic function fields.

(b) By definition, the function field of an irreducible variety X is exactly the stalk of the structure sheaf $\mathscr{O}_X$ at X in the sense of Exercise 3.23. In particular, if X is affine then this exercise shows that $K(X)$ is the localization of the coordinate ring $A(X)$ at the ideal $I(X) = \langle 0 \rangle$, i. e. the quotient field of $A(X)$.

Exercise 9.9. Show that any irreducible quadric hypersurface in $\mathbb{P}^n$ is birational, but in general not isomorphic to the projective space $\mathbb{P}^{n-1}$.

The main goal of this chapter is now to describe and study a general procedure to modify an irreducible variety to a birational one. In its original form, this construction depends on given polynomial functions $f_1, \dots, f_r$ on an affine variety X — but we will see in Construction 9.17 that it can also be performed with a given ideal in $A(X)$ or subvariety of X instead, and that it can be glued in order to work on arbitrary varieties.

Construction 9.10 (Blowing up)**.** Let $X \subset \mathbb{A}^n$ be an affine variety. For some given polynomial functions $f_1,\dots,f_r \in A(X)$ on X, we set $U = X\backslash V(f_1,\dots,f_r)$. As $f_1,\dots,f_r$ then do not vanish simultaneously at any point of U, there is a well-defined morphism

$$f\colon U \to \mathbb{P}^{r-1},\ x \mapsto (f_1(x):\dots:f_r(x)).$$

We consider its graph

$$\Gamma_f = \{(x, f(x)) : x \in U\} \quad \subset U \times \mathbb{P}^{r-1}.$$

It is closed in $U \times \mathbb{P}^{r-1}$ by Proposition 5.20 (a), but in general not in $X \times \mathbb{P}^{r-1}$. The closure of Γ_f in $X \times \mathbb{P}^{r-1}$ is called the **blow-up** of X at $f_1,\dots,f_r$; we will usually denote it by $\tilde{X}$. Note that there is a natural projection morphism $\pi\colon \tilde{X} \to X$ to the first factor. Sometimes we will also say that this morphism π is the blow-up of X at $f_1,\dots,f_r$.

Before we give examples of blow-ups let us introduce some more notation and easy general results that will help us to deal with them.

Remark 9.11 (Exceptional sets)**.** In Construction 9.10, the graph Γ_f is isomorphic to U, with isomorphism $\pi|_{\Gamma_f}\colon \Gamma_f \to U$. By abuse of notation, one often uses this isomorphism to identify Γ_f with U, so that U becomes a dense open subset of $\tilde{X}$. Its complement $\tilde{X}\backslash U = \pi^{-1}(V(f_1,\dots,f_r))$, on which π is usually not an isomorphism, is called the **exceptional set** of the blow-up.

If X is irreducible and $f_1,\dots,f_r$ do not vanish simultaneously on all of X, then $U = X\backslash V(f_1,\dots,f_r)$ is a non-empty and hence dense open subset of X. So its closure in the blow-up, which is all of $\tilde{X}$ by definition, is also irreducible. We therefore conclude that X and $\tilde{X}$ are birational in this case, with common dense open subset U.

Remark 9.12 (Strict transforms and blow-ups of subvarieties)**.** In the notation of Construction 9.10, let Y be a closed subvariety of X. Then we can blow up Y at $f_1,\dots,f_r$ as well. By construction, the resulting space $\tilde{Y} \subset Y \times \mathbb{P}^{r-1} \subset X \times \mathbb{P}^{r-1}$ is then also a closed subvariety of $\tilde{X}$, in fact it is the closure of $Y \cap U$ in $\tilde{X}$ (using the isomorphism $\Gamma_f \cong U$ of Remark 9.11 to identify $Y \cap U$ with a subset of $\tilde{X}$). If we consider $\tilde{Y}$ as a subset of $\tilde{X}$ in this way it is often called the **strict transform** of Y in the blow-up of X.

In particular, if $X = X_1 \cup \dots \cup X_m$ is the irreducible decomposition of X then $\tilde{X}_i \subset \tilde{X}$ for $i = 1,\dots,m$. Moreover, since taking closures commutes with finite unions it is immediate from Construction 9.10 that

$$\tilde{X} = \tilde{X}_1 \cup \dots \cup \tilde{X}_m,$$

i.e. that for blowing up X we just blow up its irreducible components individually. For many purposes it therefore suffices to consider blow-ups of irreducible varieties.

Example 9.13 (Trivial cases of blow-ups)**.** Let $r = 1$ in the notation of Construction 9.10, i.e. consider the case when we blow up X at only one function f_1. Then $\tilde{X} \subset X \times \mathbb{P}^0 \cong X$, and $\Gamma_f \cong U$. So $\tilde{X}$ is just the closure of U in X under this isomorphism. If we assume for simplicity that X is irreducible we therefore obtain the following two cases:

(a) If $f_1 \neq 0$ then $U = X\backslash V(f_1)$ is a non-empty open subset of X, and hence $\tilde{X} = X$ by Remark 2.16 (b).

(b) If $f_1 = 0$ then $U = \emptyset$, and hence also $\tilde{X} = \emptyset$.

So in order to obtain interesting examples of blow-ups we will have to consider cases with $r \geq 2$.

In order to understand blow-ups better, one of our main tasks has to be to find an explicit description of them that does not refer to taking closures. The following inclusion is a first step in this direction.

Lemma 9.14. *The blow-up $\tilde{X}$ of an affine variety X at $f_1,\dots,f_r \in A(X)$ satisfies*

$$\tilde{X} \subset \{(x,y) \in X \times \mathbb{P}^{r-1} : y_i f_j(x) = y_j f_i(x) \text{ for all } i,j = 1,\dots,r\}.$$

Proof. Let $U = X \backslash V(f_1,\dots,f_r)$. Then any point $(x,y) \in U \times \mathbb{P}^{r-1}$ on the graph Γ_f of the function $f: U \to \mathbb{P}^{r-1}$, $x \mapsto (f_1(x):\dots:f_r(x))$ satisfies $(y_1:\dots:y_r) = (f_1(x):\dots:f_r(x))$, which is equivalent to $y_i f_j(x) = y_j f_i(x)$ for all $i,j = 1,\dots,r$. As these equations then also have to hold on the closure $\tilde{X}$ of Γ_f, the lemma follows. □

Example 9.15 (Blow-up of $\mathbb{A}^n$ at the coordinate functions)**.** Our first non-trivial (and in fact the most important) case of a blow-up is that of $\mathbb{A}^n$ at $x_1,\dots,x_n$. This blow-up $\widetilde{\mathbb{A}^n}$ is then isomorphic to $\mathbb{A}^n$ on the open subset $U = \mathbb{A}^n \backslash V(x_1,\dots,x_n) = \mathbb{A}^n \backslash \{0\}$, and by Lemma 9.14 we have

$$\widetilde{\mathbb{A}^n} \subset \{(x,y) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : y_i x_j = y_j x_i \text{ for all } i,j = 1,\dots,n\} =: Y. \tag{1}$$

We claim that this inclusion is in fact an equality. To see this, let us consider the open subset $U_1 = \{(x,y) \in Y : y_1 \neq 0\}$ with affine coordinates $x_1,\dots,x_n,y_2,\dots,y_n$ in which we set $y_1 = 1$. Note that for given $x_1,y_2,\dots,y_n$ the equations (1) for Y then say exactly that $x_j = x_1 y_j$ for $j = 2,\dots,n$ (since this implies $y_i x_j = x_1 y_i y_j = y_j x_i$ for all i,j). Hence there is an isomorphism

$$\mathbb{A}^n \to U_1 \subset \mathbb{A}^n \times \mathbb{P}^{n-1}, \ (x_1,y_2,\dots,y_n) \mapsto ((x_1,x_1y_2,\dots,x_1y_n),(1:y_2:\dots:y_n)). \tag{2}$$

Of course, the same holds for the open subsets U_i of Y where $y_i \neq 0$ for $i = 2,\dots,n$. Hence Y can be covered by n-dimensional affine spaces. As these affine spaces intersect e. g. in the point $((1,\dots,1),(1:\dots:1))$, this means by Exercises 2.21 (b) and 2.34 (a) that Y is irreducible of dimension n as well. But as Y contains the closed subvariety $\widetilde{\mathbb{A}^n}$ which is also of dimension n by Remarks 9.6 and 9.11, we conclude that we must already have $Y = \widetilde{\mathbb{A}^n}$.

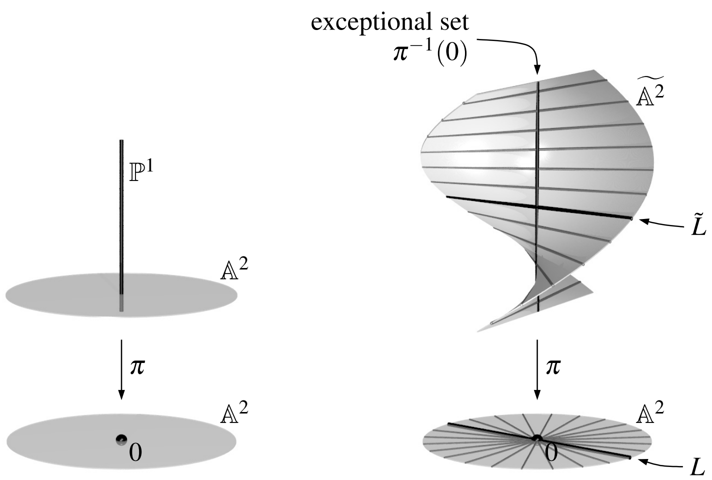
In fact, both the description (1) of $\widetilde{\mathbb{A}^n}$ (with equality, as we have just seen) and the affine coordinates of (2) are very useful in practice for explicit computations on this blow-up.

Let us now also study the blow-up (i. e. projection) morphism $\pi: \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$ of Construction 9.10. We know already that this map is an isomorphism on $U = \mathbb{A}^n \backslash \{0\}$. In contrast, the exceptional set $\pi^{-1}(0)$ is given by setting $x_1,\dots,x_n$ to 0 in the description (1) above. As all defining equations $x_i y_j = x_j y_i$ become trivial in this case, we simply get

$$\pi^{-1}(0) = \{(0,y) \in \mathbb{A}^n \times \mathbb{P}^{n-1}\} \cong \mathbb{P}^{n-1}.$$

In other words, passing from $\mathbb{A}^n$ to $\widetilde{\mathbb{A}^n}$ leaves all points except 0 unchanged, whereas the origin is replaced by a projective space $\mathbb{P}^{n-1}$. This is the geometric reason why this construction is called blowing up — in fact, we will slightly extend our terminology in Construction 9.17 (a) so that we can then call the example above the blow-up of $\mathbb{A}^n$ *at the origin*, instead of at the functions $x_1,\dots,x_n$.

Because of this behavior of the inverse images of π one might be tempted to think of $\widetilde{\mathbb{A}^n}$ as $\mathbb{A}^n$ with a projective space $\mathbb{P}^{n-1}$ attached at the origin, as in the picture below on the left. This is not correct however, as one can see already from the fact that this space would be reducible, whereas $\widetilde{\mathbb{A}^n}$ is not.



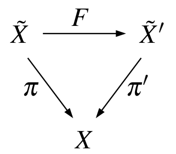
Wrong picture Correct picture

To get the true geometric picture for $\mathbb{A}^n$ let us consider the strict transform of a line $L \subset \mathbb{A}^n$ through the origin, i. e. the blow-up $\tilde{L}$ of L at $x_1, \dots, x_n$ contained in $\widetilde{\mathbb{A}^n}$. We will give a general recipe to compute such strict transforms in Exercise 9.22, but in the case at hand this can also be done without much theory: By construction, over the complement of the origin every point $(x,y) \in \tilde{L} \subset L \times \mathbb{P}^{n-1}$ must have y being equal to the projective point corresponding to $L \subset K^n$. Hence the same holds on the closure $\tilde{L}$, and thus the strict transform $\tilde{L}$ meets the exceptional set $\pi^{-1}(0) \cong \mathbb{P}^{n-1}$ exactly in the point corresponding to L. In other words, the exceptional set parametrizes the directions in $\mathbb{A}^n$ at 0; two lines through the origin with distinct directions will become separated after the blow-up. The picture above on the right illustrates this in the case of the plane: We can imagine the blow-up $\widetilde{\mathbb{A}^2}$ as a helix winding around the central line $\pi^{-1}(0) \cong \mathbb{P}^1$ (in fact, it winds around this exceptional set
15 once, so that one should think of the top of the helix as being glued to the bottom).

As already mentioned, the geometric interpretation of Example 9.15 suggests that we can think of this construction as the blow-up of $\mathbb{A}^n$ *at the origin* instead of *at the functions* $x_1, \dots, x_n$. To justify this notation let us now show that the blow-up construction does not actually depend on the chosen functions, but only on the ideal generated by them.

Lemma 9.16. *The blow-up of an affine variety X at $f_1, \dots, f_r \in A(X)$ depends only on the ideal $\langle f_1, \dots, f_r \rangle \trianglelefteq A(X)$.*

More precisely, if $f_1', \dots, f_s' \in A(X)$ with $\langle f_1, \dots, f_r \rangle = \langle f_1', \dots, f_s' \rangle \trianglelefteq A(X)$, and $\pi\colon \tilde{X} \to X$ and $\pi'\colon \tilde{X}' \to X$ are the corresponding blow-ups, there is an isomorphism $F\colon \tilde{X} \to \tilde{X}'$ with $\pi' \circ F = \pi$. In other words, we get a commutative diagram as in the picture on the right.



Proof. By assumption we have relations

$$f_i = \sum_{j=1}^{s} g_{i,j} f_j' \text{ for all } i = 1, \dots, r \quad \text{and} \quad f_j' = \sum_{k=1}^{r} h_{j,k} f_k \text{ for all } j = 1, \dots, s$$

in $A(X)$ for suitable $g_{i,j}, h_{j,k} \in A(X)$. We claim that then

$$F\colon \tilde{X} \to \tilde{X}',\ (x,y) \mapsto (x,y') := \left(x, \left(\sum_{k=1}^{r} h_{1,k}(x)\, y_k : \cdots : \sum_{k=1}^{r} h_{s,k}(x)\, y_k \right) \right)$$

is an isomorphism between $\tilde{X} \subset X \times \mathbb{P}^{r-1}$ and $\tilde{X}' \subset X \times \mathbb{P}^{s-1}$ as required. This is easy to check:

- The homogeneous coordinates of y' are not simultaneously 0: Note that by construction we have the relation $(y_1 : \cdots : y_r) = (f_1 : \cdots : f_r)$ on $U = X \backslash V(f_1, \dots, f_r) \subset \tilde{X} \subset X \times \mathbb{P}^{r-1}$, i. e. these two vectors are linearly dependent (and non-zero) at each point in this set. Hence the linear relations $f_i = \sum_{j,k} g_{i,j} h_{j,k} f_k$ in $f_1, \dots, f_r$ imply the corresponding relations $y_i = \sum_{j,k} g_{i,j} h_{j,k} y_k$ in $y_1, \dots, y_r$ on this set, and thus also on its closure $\tilde{X}$. So if we had $y_j' = \sum_k h_{j,k} y_k = 0$ for all j then we would also have $y_i = \sum_j g_{i,j} y_j' = 0$ for all i, which is a contradiction.
- The image of F lies in $\tilde{X}'$: By construction we have

$$F(x,y) = \left(x, \left(\sum_{k=1}^{r} h_{1,k}(x)\, f_k(x) : \cdots : \sum_{k=1}^{r} h_{s,k}(x)\, f_k(x) \right) \right) = \big(x, (f_1'(x) : \cdots : f_s'(x))\big) \quad \in \tilde{X}'$$

  on the open subset U, and hence also on its closure $\tilde{X}$.
- F is an isomorphism: By symmetry the same construction as above can also be done in the other direction and gives us an inverse morphism F^{-1}.
- It is obvious that $\pi' \circ F = \pi$. □

Construction 9.17 (Generalizations of the blow-up construction)**.**

(a) Let X be an affine variety. For an ideal $J \trianglelefteq A(X)$ we define the *blow-up of X at J* to be the blow-up of X at any set of generators of J — which is well-defined up to isomorphisms by

Lemma 9.16. If $Y \subset X$ is a closed subvariety the blow-up of X at $I(Y) \trianglelefteq A(X)$ will also be called the *blow-up of X at Y*. So in this language we can say that Example 9.15 describes the blow-up of $\mathbb{A}^n$ at the origin.

(b) Now let X be an arbitrary variety, and let $Y \subset X$ be a closed subvariety. For an affine open cover $\{U_i : i \in I\}$ of X, let $\tilde{U}_i$ be the blow-up of U_i at the closed subvariety $U_i \cap Y$. It is then easy to check that these blow-ups $\tilde{U}_i$ can be glued together to a variety $\tilde{X}$. We will call it again the *blow-up of X at Y*.

In the following, we will probably only need this in the case of the blow-up of a point, where the construction is even easier as it is local around the blown-up point: Let X be a variety, and let $a \in X$ be a point. Choose an affine open neighborhood $U \subset X$ of a, and let $\tilde{U}$ be the blow-up of U at a. Then we obtain $\tilde{X}$ by gluing $X \backslash \{a\}$ to $\tilde{U}$ along the common open subset $U \backslash \{a\}$.

(c) With our current techniques the gluing procedure of (b) only works for blow-ups at subvarieties — for the general construction of blowing up ideals we would need a way to patch ideals. This is in fact possible and leads to the notion of a *sheaf of ideals*, see Remark 14.10. We will not consider such blow-ups in these notes however.

Note however that *blow-ups of a projective variety X* can be defined in essentially the same way as for affine varieties: If $f_1, \dots, f_r \in S(X)$ are homogeneous of the same degree the blow-up of X at $f_1, \dots, f_r$ is defined as the closure of the graph

$$\Gamma = \{(x, (f_1(x) : \cdots : f_r(x)) : x \in U\} \quad \subset U \times \mathbb{P}^{r-1}$$

(for $U = X \backslash V_p(f_1, \dots, f_r)$) in $X \times \mathbb{P}^{r-1}$; by the Segre embedding as in Remark 7.13 it is again a projective variety.

Exercise 9.18. Let $\widetilde{\mathbb{A}^3}$ be the blow-up of $\mathbb{A}^3$ at the line $V(x_1, x_2) \cong \mathbb{A}^1$. Show that its exceptional set is isomorphic to $\mathbb{A}^1 \times \mathbb{P}^1$. When do the strict transforms of two lines in $\mathbb{A}^3$ through $V(x_1, x_2)$ intersect in the blow-up? What is therefore the geometric meaning of the points in the exceptional set (corresponding to Example 9.15 in which the points of the exceptional set correspond to the directions through the blown-up point)?

Exercise 9.19. Let $X \subset \mathbb{A}^n$ be an affine variety, and let $Y_1, Y_2 \subsetneq X$ be irreducible, closed subsets, no-one contained in the other. Moreover, let $\tilde{X}$ be the blow-up of X at the ideal $I(Y_1) + I(Y_2)$.

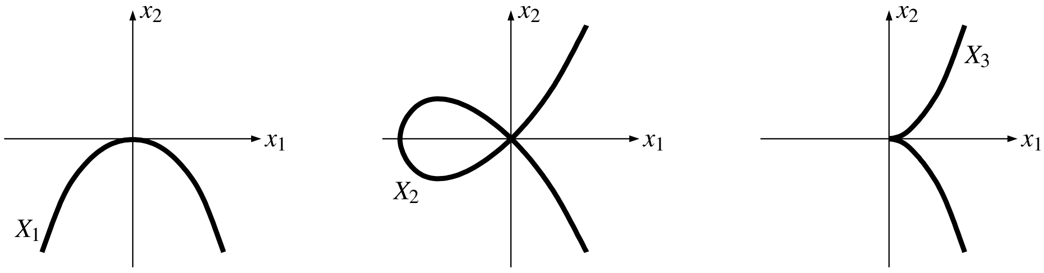
Show that the strict transforms of Y_1 and Y_2 in $\tilde{X}$ are disjoint.

One of the main applications of blow-ups is the local study of varieties. We have seen already in Example 9.15 that the exceptional set of the blow-up of $\mathbb{A}^n$ at the origin parametrizes the directions of lines at this point. It should therefore not come as a surprise that the exceptional set of the blow-up of a general variety X at a point $a \in X$ parametrizes the tangent directions of X at a.

Construction 9.20 (Tangent cones)**.** Let a be a point on a variety X. We consider the blow-up $\pi \colon \tilde{X} \to X$ of X at a; its exceptional set $\pi^{-1}(\{a\})$ is then a projective variety (e. g. by choosing an affine open neighborhood U of a in X, which is then an affine variety $U \subset \mathbb{A}^n$ with $a = (a_1, \dots, a_n)$ that we can blow up at $x_1 - a_1, \dots, x_n - a_n$; the exceptional set is then contained in the projective space $\{a\} \times \mathbb{P}^{n-1} \subset U \times \mathbb{P}^{n-1}$).

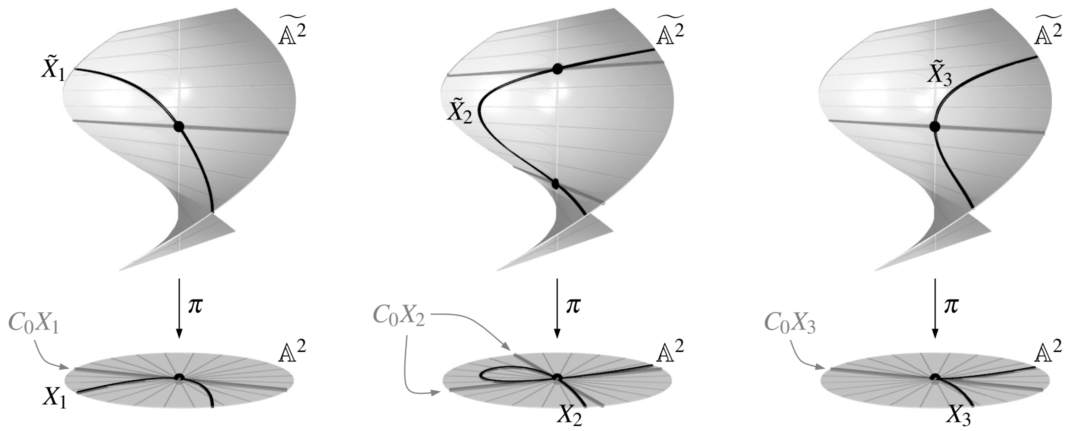
The cone over this exceptional set $\pi^{-1}(\{a\})$ (as in Definition 6.16 (c)) is called the **tangent cone** $C_a X$ of X at a. Note that it is well-defined up to isomorphism by Lemma 9.16. In the special case (of an affine patch) when $X \subset \mathbb{A}^n$ and $a \in X$ is the origin, we will also consider $C_a X \subset C(\mathbb{P}^{n-1}) = \mathbb{A}^n$ as a closed subvariety of the same ambient affine space as for X by blowing up at $x_1, \dots, x_n$.

Example 9.21. Consider the three complex affine curves $X_1, X_2, X_3 \subset \mathbb{A}^2_{\mathbb{C}}$ with real parts as in the picture below.



$X_1 = V(x_2 + x_1^2)$ $X_2 = V(x_2^2 - x_1^2 - x_1^3)$ $X_3 = V(x_2^2 - x_1^3)$

Note that by Remark 9.12 the blow-ups $\tilde{X}_i$ of these curves at the origin (for $i = 1,2,3$) are contained as strict transforms in the blow-up $\widetilde{\mathbb{A}^2}$ of the affine plane at the origin as in Example 9.15. They can thus be obtained geometrically as in the following picture by lifting the curves $X_i \backslash \{0\}$ by the map $\pi \colon \widetilde{\mathbb{A}^2} \to \mathbb{A}^2$ and taking the closure in $\widetilde{\mathbb{A}^2}$. The additional points in these closures (drawn as dots in the picture below) are the exceptional sets of the blow-ups. By definition, the tangent cones $C_0 X_i$ then consist of the lines corresponding to these points, as shown in gray below. They can be thought of as the cones, i. e. unions of lines, that approximate X_i best around the origin.



Let us now study how these tangent cones can be computed rigorously. For example, for a point $((x_1,x_2),(y_1:y_2)) \in \tilde{X}_2 \subset \widetilde{\mathbb{A}^2} \subset \mathbb{A}^2 \times \mathbb{P}^1$ we have $x_2^2 - x_1^2 - x_1^3 = 0$ (as the equation of the curve) and $y_1 x_2 - y_2 x_1 = 0$ by Lemma 9.14. The latter means that the vectors (x_1,x_2) and (y_1,y_2) are linearly dependent, i. e. that $y_1 = \lambda x_1$ and $y_2 = \lambda x_2$ away from the origin for some non-zero $\lambda \in K$. Multiplying the equation of the curve with λ^2 thus yields

$$\lambda^2 (x_2^2 - x_1^2 - x_1^3) = 0 \quad \Rightarrow \quad y_2^2 - y_1^2 - y_1^2 x_1 = 0$$

on $\tilde{X}_2 \backslash \pi^{-1}(\{0\})$, and thus also on its closure $\tilde{X}_2$. On $\pi^{-1}(\{0\})$, i. e. if $x_1 = x_2 = 0$, this implies

$$y_2^2 - y_1^2 = 0 \quad \Rightarrow \quad (y_2 - y_1)(y_2 + y_1) = 0,$$

so that the exceptional set consists of the two points with $(y_1 : y_2) \in \mathbb{P}^1$ equal to $(1:1)$ or $(1:-1)$. Consequently, the tangent cone $C_0 X_2$ is the cone in $\mathbb{A}^2$ with the same equation

$$(x_2 - x_1)(x_2 + x_1) = 0,$$

i. e. the union of the two diagonals in $\mathbb{A}^2$ as in the picture above.

Note that the effect of this computation was exactly to pick out the terms of minimal degree of the defining equation $x_2^2 - x_1^2 - x_1^3 = 0$ — in this case of degree 2 — to obtain the equation $x_2^2 - x_1^2 = 0$ of the tangent cone at the origin. This obviously yields a homogeneous polynomial (so that its affine zero locus is a cone), and it fits well with the intuitive idea that for small values of x_1 and x_2 the higher powers of the coordinates are much smaller, so that we get a good approximation for the curve around the origin when we neglect them.

In fact, the following exercise (which is similar in style to proposition 6.32) shows that taking the terms of smallest degree of the defining equations is the general way to compute tangent cones explicitly after the coordinates have been shifted so that the point under consideration is the origin.

Exercise 9.22 (Computation of tangent cones)**.** Let $J \trianglelefteq K[x_1,\dots,x_n]$ be an ideal, and assume that the affine variety $X = V(J) \subset \mathbb{A}^n$ contains the origin. Consider the blow-up $\tilde{X} \subset \widetilde{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ at $x_1,\dots,x_n$, and denote the homogeneous coordinates of $\mathbb{P}^{n-1}$ by $y_1,\dots,y_n$.

(a) By Example 9.15 we know that $\widetilde{\mathbb{A}^n}$ can be covered by affine spaces, with one coordinate patch being

$$\mathbb{A}^n \to \widetilde{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{P}^{n-1},$$
$$(x_1,y_2,\dots,y_n) \mapsto ((x_1,x_1y_2,\dots,x_1y_n),(1:y_2:\dots:y_n)).$$

Prove that on this coordinate patch the blow-up $\tilde{X}$ is given as the zero locus of the polynomials

$$\frac{f(x_1,x_1y_2,\dots,x_1y_n)}{x_1^{\min\deg f}}$$

for all non-zero $f \in J$, where $\min\deg f$ denotes the smallest degree of a monomial in f.

(b) Prove that the exceptional set of $\tilde{X}$ is

$$V_p(f^{\text{in}} : f \in J) \quad \subset \{0\} \times \mathbb{P}^{n-1},$$

where f^{in} is the *initial term* of f, i. e. the sum of all monomials in f of smallest degree. Consequently, the tangent cone of X at the origin is

$$C_0X = V_a(f^{\text{in}} : f \in J) \quad \subset \mathbb{A}^n.$$

(c) If $J = \langle f \rangle$ is a principal ideal prove that $C_0X = V_a(f^{\text{in}})$. However, for a general ideal J show that C_0X is in general not the zero locus of the initial terms of a set of generators for J.

In Example 9.15 above, blowing up the n-dimensional variety $\mathbb{A}^n$ at $x_1,\dots,x_n$ has replaced the origin by a variety $\mathbb{P}^{n-1}$ of codimension 1 in $\widetilde{\mathbb{A}^n}$. We will now see that this is in fact a general phenomenon.

Proposition 9.23 (Dimension of the exceptional set)**.** *Let $\pi\colon \tilde{X} \to X$ be the blow-up of an irreducible affine variety X at $f_1,\dots,f_r \in A(X)$. Then every irreducible component of the exceptional set $\pi^{-1}(V(f_1,\dots,f_r))$ has codimension 1 in $\tilde{X}$. It is therefore often called the* ***exceptional hypersurface*** *of the blow-up.*

Proof. It is enough to prove the statement on all non-empty affine open subsets $U_i \subset \tilde{X} \subset X \times \mathbb{P}^{r-1}$ where the i-th projective coordinate y_i is non-zero, since these open subsets cover $\tilde{X}$. But note that for $a \in U_i$ the condition $f_i(a) = 0$ implies $f_j(a) = 0$ for all j by Lemma 9.14. So the exceptional set is given by one equation $f_i = 0$ on U_i. Moreover, if U_i is non-empty then this polynomial f_i is not identically zero on U_i: Otherwise U_i, and thus also its closure $\tilde{X}$, would be contained in the exceptional set — which is a contradiction since this implies $U = \emptyset$ and thus $\tilde{X} = \emptyset$. The statement of the lemma thus follows from Proposition 2.28 (c). □

Corollary 9.24 (Dimension of tangent cones)**.** *Let a be a point on a variety X. Then the dimension* $\dim C_aX$ *of the tangent cone of X at a is the local dimension* $\operatorname{codim}_X\{a\}$ *of X at a.*

Proof. Note that both $\dim C_aX$ and $\operatorname{codim}_X\{a\}$ are local around the point a. By passing to an open neighborhood of a we can therefore assume that every irreducible component of X meets a, and that $X \subset \mathbb{A}^n$ is affine. We may also assume that X is not just the one-point set $\{a\}$, since otherwise the statement of the corollary is trivial.

Now let $X = X_1 \cup \dots \cup X_m$ be the irreducible decomposition of X. Note that $X \neq \{a\}$ implies that all of these components have dimension at least 1. By Proposition 9.23 every irreducible component of the exceptional set of the blow-up $\tilde{X}_i$ of X_i at a has dimension $\dim X_i - 1$, and so by Exercise 6.31 (a) every irreducible component of the tangent cone $C_a X_i$ has dimension $\dim X_i$. As the maximum of these dimensions is just the local dimension $\operatorname{codim}_X\{a\}$ (see Exercise 2.35) it therefore suffices to show that all these exceptional sets are non-empty.

Assume the contrary, i. e. that the exceptional set of $\tilde{X}_i$ is empty for some i. Extending this to the projective closure $\mathbb{P}^n$ of $\mathbb{A}^n$ we obtain an irreducible variety $\overline{X_i} \subset \mathbb{P}^n$ containing a whose blow-up $\widetilde{\overline{X_i}}$ in $\widetilde{\mathbb{P}^n}$ has an empty exceptional set. This means that $\pi\Big(\widetilde{\overline{X_i}}\Big) = \overline{X_i} \backslash \{a\}$, where $\pi\colon \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$ is the blow-up map. As $\widetilde{\overline{X_i}}$ is a projective (and hence complete) variety by Construction 9.17 (c) this is a contradiction to Corollary 7.23 since $\overline{X_i}\backslash\{a\}$ is not closed (recall that $\overline{X_i}$ has dimension at least 1, so that $\overline{X_i}\backslash\{a\} \neq \emptyset$). □

Exercise 9.25.

(a) Show that the blow-up of $\mathbb{A}^2$ at the ideal $\langle x_1^2, x_1x_2, x_2^2 \rangle$ is isomorphic to the blow-up of $\mathbb{A}^2$ at the ideal $\langle x_1, x_2 \rangle$.

(b) Let X be an affine variety, and let $J \trianglelefteq A(X)$ be an ideal. Show by example that the blow-up of X at J is in general not isomorphic to the blow-up of X at $\sqrt{J}$.

(Hint: Tangent cones are invariant under isomorphisms.)

We will now discuss another important application of blow-ups that follows more or less directly from the definitions: They can be used to extend morphisms defined only on an open subset of a variety.

Remark 9.26 (Blowing up to extend morphisms)**.** Let X be an affine variety, and let $f_1, \dots, f_r$ be polynomial functions on X. Note that the morphism $f\colon x \mapsto (f_1(x) : \cdots : f_r(x))$ to $\mathbb{P}^{r-1}$ is only well-defined on the open subset $U = X\backslash V(f_1, \dots, f_r)$ of X. In general, we can not expect that this morphism can be extended to a morphism on all of X. But we can always extend it "after blowing up the ideal $\langle f_1, \dots, f_r \rangle$ of the indeterminacy locus": There is an extension $\tilde{f}\colon \tilde{X} \to \mathbb{P}^{r-1}$ of f (that agrees with f on U), namely just the projection from $\tilde{X} \subset X \times \mathbb{P}^{r-1}$ to the second factor $\mathbb{P}^{r-1}$. So blowing up is a way to extend morphisms to bigger sets on which they would otherwise be ill-defined. Let us consider a concrete example of this idea in the next lemma and the following remark.

Lemma 9.27. *$\mathbb{P}^1 \times \mathbb{P}^1$ blown up in one point is isomorphic to $\mathbb{P}^2$ blown up in two points.*

Proof. We know from Example 7.11 that $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the quadric surface

$$X = \{(x_0 : x_1 : x_2 : x_3) : x_0x_3 = x_1x_2\} \quad \subset \mathbb{P}^3.$$

Let $\tilde{X}$ be blow-up of X at $a = (0:0:0:1) \in X$, which can be realized as in Construction 9.17 (c) as the blow-up $\tilde{X} \subset \mathbb{P}^3 \times \mathbb{P}^2$ of X at x_0, x_1, x_2.

On the other hand, let $b = (0:1:0), c = (0:0:1) \in \mathbb{P}^2$, and let $\widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^3$ be the blow-up of $\mathbb{P}^2$ at $y_0^2, y_0y_1, y_0y_2, y_1y_2$ as in Construction 9.17 (c). Note that these polynomials do not generate the ideal $I(\{b,c\}) = \langle y_0, y_1y_2 \rangle$, but this does not matter: The blow-up is a local construction, so let us check that we are locally just blowing up b, and similarly c. There is an open affine neighborhood around b given by $y_1 \neq 0$, where we can set $y_1 = 1$, and on this neighborhood the given functions y_0^2, y_0, y_0y_2, y_2 generate the ideal $\langle y_0, y_2 \rangle$ of b. So $\widetilde{\mathbb{P}^2}$ is actually the blow-up of $\mathbb{P}^2$ at b and c.

Now we claim that an isomorphism is given by

$$f\colon \tilde{X} \mapsto \widetilde{\mathbb{P}^2},\ ((x_0 : x_1 : x_2 : x_3), (y_0 : y_1 : y_2)) \mapsto ((y_0 : y_1 : y_2), (x_0 : x_1 : x_2 : x_3)).$$

In fact, this is easy to prove: Obviously, f is an isomorphism from $\mathbb{P}^3 \times \mathbb{P}^2$ to $\mathbb{P}^2 \times \mathbb{P}^3$, so we only have to show that f maps $\tilde{X}$ to $\widetilde{\mathbb{P}^2}$, and that f^{-1} maps $\widetilde{\mathbb{P}^2}$ to $\tilde{X}$. Note that it suffices to check this on a dense open subset. But this is easy: On the complement of the exceptional set in $\tilde{X}$ we have $x_0x_3 = x_1x_2$ and $(y_0 : y_1 : y_2) = (x_0 : x_1 : x_2)$, so on the (smaller) complement of $V(x_0)$ we get the correct equations

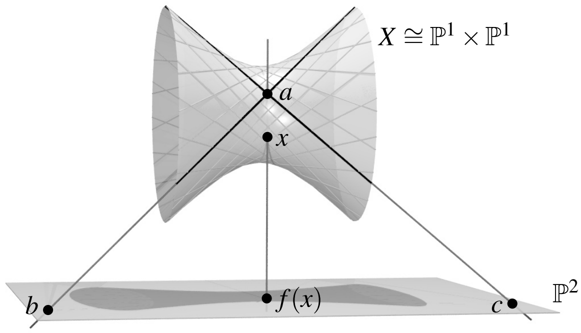
$$(x_0 : x_1 : x_2 : x_3) = (x_0^2 : x_0x_1 : x_0x_2 : x_0x_3) = (x_0^2 : x_0x_1 : x_0x_2 : x_1x_2) = (y_0^2 : y_0y_1 : y_0y_2 : y_1y_2)$$

for the image point under f to lie in $\widetilde{\mathbb{P}^2}$. Conversely, on the complement of the exceptional hypersurface in $\widetilde{\mathbb{P}^2}$ we have $(x_0:x_1:x_2:x_3) = (y_0^2:y_0y_1:y_0y_2:y_1y_2)$, so we conclude that $x_0x_3 = x_1x_2$ and $(y_0:y_1:y_2) = (x_0:x_1:x_2)$ where $y_0 \neq 0$. □

Remark 9.28. The proof of Lemma 9.27 is short and elegant, but not very insightful. So let us try to understand geometrically what is going on. As in the proof above, we think of $\mathbb{P}^1 \times \mathbb{P}^1$ as the quadric surface

$$X = \{(x_0:x_1:x_2:x_3) : x_0x_3 = x_1x_2\} \quad \subset \mathbb{P}^3.$$

Let us project X from $a = (0:0:0:1) \in X$ to $V_p(x_3) \cong \mathbb{P}^2$. The corresponding morphism f is shown in the picture below; as in Example 7.5 (b) it is given by $f(x_0:x_1:x_2:x_3) = (x_0:x_1:x_2)$ and well-defined away from a.



Recall that, in the corresponding case of the projection of a quadric *curve* in Example 7.5 (c), the morphism f could be extended to the point a. This is now no longer the case for our quadric *surface* X: To construct $f(a)$ we would have to take the limit of the points $f(x)$ for x approaching a, i. e. consider lines through a and x for $x \to a$. These lines will then become tangent lines to X at a — but X, being two-dimensional, has a one-parameter family of such tangent lines. This is why $f(a)$ is ill-defined. But we also see from this discussion that blowing up a on X, i. e. replacing it by the set of all tangent lines through a, will exactly resolve this indeterminacy. Hence f becomes a well-defined morphism from $\tilde{X}$ to $V_p(x_3) \cong \mathbb{P}^2$.

Let us now check if there is an inverse morphism. By construction, it is easy to see what it would have to look like: The points of $X \backslash \{a\}$ mapped to a point $y \in V_p(x_3)$ are exactly those on the line $\overline{ay}$ through a and y. In general, this line intersects X in two points, one of which is a. So there is then exactly one point on X which maps to y, leading to an inverse morphism f^{-1}. This reasoning is only false if the whole line $\overline{ay}$ lies in X. Then this whole line would be mapped to y, so that we cannot have an inverse f^{-1} there. But of course we expect again that this problem can be taken care of by blowing up y in $\mathbb{P}^2$, so that it is replaced by a $\mathbb{P}^1$ that can then be mapped bijectively to $\overline{ay}$.

There are obviously two such lines $\overline{ab}$ and $\overline{ac}$, given by $b = (0:1:0)$ and $c = (0:0:1)$. If you think of X as $\mathbb{P}^1 \times \mathbb{P}^1$ again, these lines are precisely the "horizontal" and "vertical" lines passing through a where the coordinate in one of the two factors is constant. So we would expect that f can be made into an isomorphism after blowing up b and c, which is exactly what we have shown in Lemma 9.27.

Exercise 9.29 (Cremona transformation)**.** Let $a = (1:0:0)$, $b = (0:1:0)$, and $c = (0:0:1)$ be the three coordinate points of $\mathbb{P}^2$, and let $U = \mathbb{P}^2 \backslash \{a,b,c\}$. Consider the morphism

$$f\colon U \to \mathbb{P}^2,\ (x_0:x_1:x_2) \mapsto (x_1x_2:x_0x_2:x_0x_1).$$

(a) Show that there is no morphism $\mathbb{P}^2 \to \mathbb{P}^2$ extending f.

(b) Let $\widetilde{\mathbb{P}^2}$ be the blow-up of $\mathbb{P}^2$ at $\{a,b,c\}$. Show that f can be extended to an isomorphism $\tilde{f}\colon \widetilde{\mathbb{P}^2} \to \widetilde{\mathbb{P}^2}$. This isomorphism is called the *Cremona transformation*.

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