2. The Zariski Topology

In this chapter we will define a *topology* on an affine variety X, i. e. a notion of open and closed subsets of X. We will see that many properties of X can be expressed purely in terms of this topology, e. g. its dimension or the question whether it consists of several components. The advantage of this is that these concepts can then easily be reused later in Chapter 5 when we consider more general varieties — which are still topological spaces, but arise in a slightly different way.

Compared to e. g. the standard topology on subsets of real vector spaces, the properties of our topology on affine varieties will be very unusual. Consequently, most concepts and results covered in a standard introductory course on topology will be trivial or useless in our case, so that we will only need the very first definitions of general topology. Let us quickly review them here.

Remark 2.1 (Topologies)**.** A *topology* on a set X is given by declaring some subsets of X to be *closed*, such that the following properties hold:

(a) the empty set $\emptyset$ and the whole space X are closed;

(b) arbitrary intersections of closed sets are closed;

(c) finite unions of closed sets are closed.

Given such a topology on X, a subset U of X is then called *open* if its complement $X\backslash U$ is closed. The *closure* $\overline{A}$ of a subset $A \subset X$ is defined to be the smallest closed subset containing A, or more precisely the intersection of all closed subsets containing A (which is closed again by (b)).

A topology on X induces a *subspace topology* on any subset $A \subset X$ by declaring the subsets of A to be closed that are of the form $A \cap Y$ for a closed subset Y of X (or equivalently the subsets of A to be open that are of the form $A \cap U$ for an open subset U of X). Subsets of topological spaces will always be equipped with this subspace topology unless stated otherwise. Note that if A is closed itself then the closed subsets of A in the subspace topology are exactly the closed subsets of X contained in A; if A is open then the open subsets of A in the subspace topology are exactly the open subsets of X contained in A.

A map $f\colon X \to Y$ between topological spaces is called *continuous* if inverse images of closed subsets of Y under f are closed in X, or equivalently if inverse images of open subsets are open.

Note that the standard definition of closed subsets in $\mathbb{R}^n$ (or more generally in metric spaces) that you know from real analysis satisfies the conditions (a), (b), and (c), and leads with the above definitions to the well-known notions of open subsets, closures, and continuous functions.

With these preparations we can now define the standard topology used in algebraic geometry.

Definition 2.2 (Zariski topology)**.** Let X be an affine variety. We define the **Zariski topology** on X to be the topology whose closed sets are exactly the affine subvarieties of X, i. e. the subsets of the form $V(S)$ for some $S \subset A(X)$. Note that this is in fact a topology by (the relative version of) Lemma 1.4 and Example 1.5 (a).

Unless stated otherwise, topological notions for affine varieties (and their subsets, using the subspace topology of Remark 2.1) will always be understood with respect to this topology.

Remark 2.3. Let $X \subset \mathbb{A}^n$ be an affine variety. Given Remark 2.1 and Definition 2.2, it seems that we have just defined several topologies on X:

(a) the Zariski topology on X, whose closed subsets are the affine subvarieties of X; and

(b) the subspace topology of X in any fixed affine variety $Y \subset \mathbb{A}^n$ with $X \subset Y$ using the Zariski topology on Y, i. e. the topology whose closed subsets are the sets of the form $X \cap Z$, with Z an affine subvariety of Y.

But in fact all these topologies agree, since the affine subvarieties of X are precisely the affine varieties contained in X, and the intersection of two affine varieties is again an affine variety. Hence it will not lead to confusion if we consider all these topologies to be the standard topology on X.

Example 2.4 (Topologies on complex varieties)**.** Compared to the classical metric topology in the case of the ground field $\mathbb{C}$, the Zariski topology is certainly unusual:

(a) The metric unit ball $A = \{x \in \mathbb{A}^1_{\mathbb{C}} : |x| \leq 1\}$ in $\mathbb{A}^1_{\mathbb{C}}$ is clearly closed in the classical topology, but not in the Zariski topology. In fact, by Example 1.5 (b) the Zariski-closed subsets of $\mathbb{A}^1$ are only the finite sets and $\mathbb{A}^1$ itself. In particular, the closure of A in the Zariski topology is all of $\mathbb{A}^1$.

Intuitively, we can say that the closed subsets in the Zariski topology are very "small", and hence that the open subsets are very "big" (see also Remark 2.16). Any Zariski-closed subset is also closed in the classical topology (since it is given by equations among polynomial functions, which are continuous in the classical topology), but as the above example shows only "very few" closed subsets in the classical topology are also Zariski-closed.

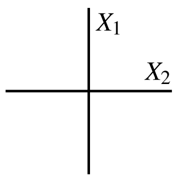
(b) Let $f \colon \mathbb{A}^1 \to \mathbb{A}^1$ be *any* injective map. Then f is automatically continuous in the Zariski topology by Example 1.5 (b), since inverse images of finite subsets of $\mathbb{A}^1$ under f are again finite.

This statement is essentially useless however, as we will not define morphisms of affine varieties as just being continuous maps with respect to the Zariski topology. In fact, this example gives us a strong hint that we should not do so.

(c) In general topology there is a notion of a *product topology*: If X and Y are topological spaces then $X \times Y$ has a natural structure of a topological space by saying that a subset of $X \times Y$ is open if and only if it is a union of products $U_i \times V_i$ for open subsets $U_i \subset X$ and $V_i \subset Y$ with i in an arbitrary index set.

With this construction, note however that the Zariski topology of an affine product variety $X \times Y$ is not the product topology: For example, the subset $V(x_1 - x_2) = \{(a,a) : a \in K\} \subset \mathbb{A}^2$ is closed in the Zariski topology, but not in the product topology of $\mathbb{A}^1 \times \mathbb{A}^1$. In fact, we will see in Proposition 4.10 that the Zariski topology is the "correct" one, whereas the product topology is useless in algebraic geometry.

But let us now start with the discussion of the topological concepts that are actually useful in the Zariski topology. The first ones concern *components* of an affine variety: The affine variety $X = V(x_1 x_2) \subset \mathbb{A}^2$ as in the picture on the right can be written as the union of the two coordinate axes $X_1 = V(x_1)$ and $X_2 = V(x_2)$, which are themselves affine varieties. However, X_1 and X_2 cannot be decomposed further into finite unions of smaller affine varieties. The following notion generalizes this idea.



$X = X_1 \cup X_2$

Definition 2.5 (Irreducible and connected spaces)**.** Let X be a topological space.

(a) We say that X is **reducible** if it can be written as $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$. Otherwise X is called **irreducible**.

(b) The space X is called **disconnected** if it can be written as $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$ with $X_1 \cap X_2 = \emptyset$. Otherwise X is called **connected**.

Remark 2.6. Although we have given this definition for arbitrary topological spaces, we will usually want to apply the notion of irreducibility only in the Zariski topology. For example, in the classical topology, the complex plane $\mathbb{A}^1_{\mathbb{C}}$ is reducible because it can be written as a union of closed subsets e. g. as

$$\mathbb{A}^1_{\mathbb{C}} = \{x \in \mathbb{C} : |x| \leq 1\} \cup \{x \in \mathbb{C} : |x| \geq 1\}.$$

In the Zariski topology however, $\mathbb{A}^1$ is irreducible by Example 1.5 (b) (as it should be).

In contrast, the notion of connectedness can be used in the "usual" topology as well and does mean there what you think it should mean.

In the Zariski topology, the algebraic characterization of irreducibility and connectedness is the following.

Proposition 2.7. *Let X be a disconnected affine variety, with $X = X_1 \cup X_2$ for two disjoint closed subsets $X_1, X_2 \subsetneq X$. Then $A(X) \cong A(X_1) \times A(X_2)$.*

Proof. As $X_1 \cup X_2 = X$ we obtain in $A(X)$

$$I(X_1) \cap I(X_2) \overset{1.18\,(c)}{=} I(X_1 \cup X_2) = I(X) = \{0\}.$$

On the other hand, from $X_1 \cap X_2 = \emptyset$ we have in $A(X)$

$$\sqrt{I(X_1) + I(X_2)} \overset{1.18\,(c)}{=} I(X_1 \cap X_2) = I(\emptyset) = \langle 1 \rangle,$$

and thus also $I(X_1) + I(X_2) = \langle 1 \rangle$. So by the Chinese Remainder Theorem [G3, Proposition 1.14] we conclude that

$$A(X) \cong A(X)/I(X_1) \times A(X)/I(X_2),$$

which by Remark 1.18 (a) is exactly the statement of the proposition. □

Proposition 2.8. *A non-empty affine variety X is irreducible if and only if $A(X)$ is an integral domain.*

Proof. As X is non-empty, its coordinate ring $A(X)$ is not the zero ring (which by definition is not an integral domain).

"⇒": Assume that $A(X)$ is not an integral domain, i. e. there are non-zero $f_1, f_2 \in A(X)$ such that $f_1 f_2 = 0$. Then $X_1 = V(f_1)$ and $X_2 = V(f_2)$ are closed, not equal to X (since f_1 and f_2 are non-zero), and $X_1 \cup X_2 = V(f_1) \cup V(f_2) = V(f_1 f_2) = V(0) = X$. Hence X is reducible.

"⇐": Assume that X is reducible, with $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$. By the bijection of the relative Nullstellensatz as in Remark 1.18 (b) this means that $I(X_i) \neq \{0\}$ in $A(X)$ for $i = 1, 2$, and so we can choose non-zero $f_i \in I(X_i)$. Then $f_1 f_2$ vanishes on $X_1 \cup X_2 = X$.
02 Hence $f_1 f_2 = 0 \in A(X)$, i. e. $A(X)$ is not an integral domain. □

Remark 2.9 (Irreducible subvarieties ↔ prime ideals)**.** Let Y be an affine variety. By Remark 1.18 (a) we have $A(X) \cong A(Y)/I_Y(X)$ for any affine subvariety X of Y, and this ring is an integral domain if and only if $I_Y(X)$ is a prime ideal. Hence, by Proposition 2.8 the bijection of the relative Nullstellensatz of Remark 1.18 (b) restricts to a bijection

$$\{\text{non-empty irreducible affine subvarieties of } Y\} \quad \overset{1:1}{\longleftrightarrow} \quad \{\text{prime ideals in } A(Y)\}$$

for any affine variety Y.

Example 2.10.

(a) A finite affine variety is irreducible if and only if it is connected: namely if and only if it contains at most one point.

(b) Any irreducible space is connected.

(c) The affine space $\mathbb{A}^n$ is irreducible (and thus connected) by Proposition 2.8 since its coordinate ring $A(\mathbb{A}^n) = K[x_1, \dots, x_n]$ is an integral domain. More generally, this holds for any affine variety given by linear equations, since again its coordinate ring is isomorphic to a polynomial ring.

(d) The union $X = V(x_1 x_2) \subset \mathbb{A}^2$ of the two coordinate axes $X_1 = V(x_1)$ and $X_2 = V(x_2)$ is not irreducible, since $X = X_1 \cup X_2$. But X_1 and X_2 themselves are irreducible by (c). Hence we have decomposed X into a union of two irreducible spaces.

As already announced, we now want to see that such a decomposition into finitely many irreducible spaces is possible for any affine variety. In fact, this works for a much larger class of topological spaces, the so-called *Noetherian* spaces:

Definition 2.11 (Noetherian topological spaces)**.** A topological space X is called **Noetherian** if there is no infinite strictly decreasing chain

$$X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$$

of closed subsets of X.

Lemma 2.12. *Any affine variety is a Noetherian topological space.*

Proof. Let X be an affine variety. By the relative Nullstellensatz of Remark 1.18 (b) an infinite decreasing chain $X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$ of affine subvarieties of X would give rise to an infinite increasing chain $I(X_0) \subsetneq I(X_1) \subsetneq \cdots$ of ideals in $A(X)$, which does not exist since $A(X)$ is Noetherian [G3, Corollary 7.14]. Hence X is a Noetherian topological space. □

Remark 2.13 (Subspaces of Noetherian spaces are Noetherian)**.** Let A be a subset of a Noetherian topological space X. Then A is also Noetherian: Otherwise we would have an infinite strictly descending chain of closed subsets of A, which by definition of the subspace topology we can write as

$$A \cap X_0 \supsetneq A \cap X_1 \supsetneq A \cap X_2 \supsetneq \cdots$$

for closed subsets $X_0, X_1, X_2, \ldots$ of X. Then

$$X_0 \supset X_0 \cap X_1 \supset X_0 \cap X_1 \cap X_2 \supset \cdots$$

is a decreasing chain of closed subsets of X. In fact, in contradiction to our assumption it is also strictly decreasing, since $X_0 \cap \cdots \cap X_k = X_0 \cap \cdots \cap X_{k+1}$ for some $k \in \mathbb{N}$ would imply $A \cap X_k = A \cap X_{k+1}$ by intersecting with A.

Combining Lemma 2.12 with Remark 2.13 we therefore see that any subset of an affine variety is a Noetherian topological space. In fact, all topological spaces occurring in this class will be Noetherian, and thus we can safely restrict our attention to this class of spaces.

Proposition 2.14 (Irreducible decomposition of Noetherian spaces)**.** *Every Noetherian topological space X can be written as a finite union $X = X_1 \cup \cdots \cup X_r$ of non-empty irreducible closed subsets. If one assumes that $X_i \not\subset X_j$ for all $i \neq j$, then $X_1, \ldots, X_r$ are unique (up to permutation). They are called the* ***irreducible components*** *of X.*

Proof. For $X = \emptyset$ the statement is obvious (with $r = 0$).

Otherwise, to prove existence, assume that there is a topological space X for which the statement is false. In particular, X is reducible, hence $X = X_1 \cup X_1'$ as in Definition 2.5 (a). Moreover, the statement of the proposition must be false for at least one of these two subsets, say X_1. Continuing this construction, one arrives at an infinite chain $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$ of closed subsets, which is a contradiction as X is Noetherian.

To show uniqueness, assume that we have two decompositions

$$X = X_1 \cup \cdots \cup X_r = X_1' \cup \cdots \cup X_s'. \qquad (*)$$

Then for any fixed $i \in \{1, \ldots, r\}$ we have $X_i \subset \bigcup_j X_j'$, so $X_i = \bigcup_j (X_i \cap X_j')$. But X_i is irreducible, and so we must have $X_i = X_i \cap X_j'$, i. e. $X_i \subset X_j'$ for some j. In the same way we conclude that $X_j' \subset X_k$ for some k, so that $X_i \subset X_j' \subset X_k$. By assumption this is only possible for $i = k$, and consequently $X_i = X_j'$. Hence every set appearing on the left side of $(*)$ also appears on the right side (and vice versa), which means that the two decompositions agree. □

Remark 2.15 (Irreducible decomposition of affine varieties)**.** The irreducible decomposition of an affine variety $X \subset \mathbb{A}^n$ can be computed from the *primary decomposition* of its ideal: If

$$I(X) = Q_1 \cap \cdots \cap Q_r \quad \trianglelefteq K[x_1, \ldots, x_n]$$

is a primary decomposition of $I(X)$ with primary ideals $Q_1, \ldots, Q_r$ as in [G3, Chapter 8], we obtain by Hilbert's Nullstellensatz

$$X = V(I(X)) \overset{1.7\,(b)}{=} V(Q_1) \cup \cdots \cup V(Q_r) \overset{1.7\,(a)}{=} V(P_1) \cup \cdots \cup V(P_r)$$

for the prime ideals $P_i = \sqrt{Q_i}$ for $i = 1, \dots, r$. Note that all varieties $V(P_i)$ in this union are irreducible by Remark 2.9. So keeping only the minimal prime ideals, i. e. the maximal varieties, among them, we obtain the irreducible decomposition of X as in Proposition 2.14. Note that they correspond exactly to the minimal prime ideals in $A(X)$, so that we obtain the following additional bijection from the relative Nullstellensatz of Remark 1.18 (b):

$$\{\text{irreducible components of } X\} \quad \stackrel{1:1}{\longleftrightarrow} \quad \{\text{minimal prime ideals in } A(X)\}.$$

Remark 2.16 (Open subsets of irreducible spaces are dense)**.** We have already seen in Example 2.4 (a) that open subsets tend to be very “big” in the Zariski topology. Here are two precise statements along these lines. Let X be an irreducible topological space, and let U and U' be non-empty open subsets of X. Then:

(a) The intersection $U \cap U'$ is never empty. In fact, by taking complements this is just equivalent to saying that the union of the two proper closed subsets $X \backslash U$ and $X \backslash U'$ is not equal to X, i. e. to the definition of irreducibility.

(b) The closure $\overline{U}$ of U is all of X — one says that U is *dense* in X. This is easily seen: If $Y \subset X$ is any closed subset containing U then $X = Y \cup (X \backslash U)$, and since X is irreducible and $X \backslash U \neq X$ we must have $Y = X$.

Exercise 2.17. Find the irreducible components of the affine variety $V(x_1 - x_2 x_3, x_1 x_3 - x_2^2) \subset \mathbb{A}^3_{\mathbb{C}}$.

Exercise 2.18. Let $X \subset \mathbb{A}^n$ be an arbitrary subset. Prove that $V(I(X)) = \overline{X}$.

Exercise 2.19. Let X be a topological space.

(a) If X is Noetherian show that we can write it as a finite disjoint union $X = X_1 \cup \cdots \cup X_r$ of non-empty connected closed subsets of X, and that this decomposition is unique up to permutation. We call $X_1, \dots, X_r$ the *connected components* of X.

(b) Let $X_1, \dots, X_r$ be any subsets of X with $X = X_1 \cup \cdots \cup X_r$. If all $X_1, \dots, X_r$ are Noetherian, prove that X is Noetherian as well.

Exercise 2.20. Let A be a subset of a topological space X. Prove:

(a) If $Y \subset A$ is closed in the subspace topology of A then $\overline{Y} \cap A = Y$.

(b) A is irreducible if and only if $\overline{A}$ is irreducible.

Exercise 2.21. Let $\{U_i : i \in I\}$ be an open cover of a topological space X, and assume that $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$. Show:

(a) If U_i is connected for all $i \in I$ then X is connected.

(b) If U_i is irreducible for all $i \in I$ then X is irreducible.

Exercise 2.22. Let $f \colon X \to Y$ be a continuous map of topological spaces. Prove:

(a) If X is connected then so is $f(X)$.

(b) If X is irreducible then so is $f(X)$.

Exercise 2.23. Recall that for two ideals J_1 and J_2 in a ring R the *ideal quotient* is defined by

$$J_1 : J_2 = \{f \in R : f J_2 \subset J_1\}.$$

Show that ideal quotients correspond to differences of varieties in the following sense: If X is an affine variety and ...

(a) Y_1 and Y_2 are subvarieties of X then $I(\overline{Y_1 \backslash Y_2}) = I(Y_1) : I(Y_2)$ in $A(X)$;

(b) J_1 and J_2 are radical ideals in $A(X)$ then $\overline{V(J_1) \backslash V(J_2)} = V(J_1 : J_2)$.

Exercise 2.24. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be irreducible affine varieties. Prove that their product $X \times Y \subset \mathbb{A}^{n+m}$ is irreducible as well.

An important application of the notion of irreducibility is the definition of the dimension of an affine variety (or more generally of a topological space — but as with our other concepts above you will only want to apply it to the Zariski topology). Of course, at least in the case of complex varieties we have a geometric idea what the dimension of an affine variety should be: the number of coordinates that you need to describe X locally around any point. Although there are algebraic definitions of dimension that mimic this intuitive one [G3, Proposition 11.31], the standard definition of dimension that we will give here uses only the language of topological spaces. Its idea is simply that, if X is an *irreducible* topological space, then any closed subset of X not equal to X should have smaller dimension. The resulting definition is the following.

Definition 2.25 (Dimension and codimension)**.** Let X be a non-empty topological space.

(a) The **dimension** $\dim X \in \mathbb{N} \cup \{\infty\}$ is the supremum over all $n \in \mathbb{N}$ such that there is a chain

$$\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subset X$$

of length n of irreducible closed subsets $Y_1, \ldots, Y_n$ of X.

(b) If $Y \subset X$ is a non-empty irreducible closed subset of X the **codimension** $\operatorname{codim}_X Y$ of Y in X is again the supremum over all n such that there is a chain

$$Y \subset Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subset X$$

of irreducible closed subsets $Y_1, \ldots, Y_n$ of X containing Y.

To avoid confusion, we will always denote the dimension of a K-vector space V by $\dim_K V$, leaving the notation $\dim X$ (without an index) for the dimension of a topological space X as above.

According to the above idea, one should imagine each Y_i as having dimension i in a maximal chain as in Definition 2.25 (a), so that finally $\dim X = n$. In the same way, each Y_i in Definition 2.25 (b) should have dimension $i + \dim Y$ in a maximal chain, so that $n = \dim X - \dim Y$ should be the difference of the dimensions of X and Y.

Example 2.26.

(a) By Example 1.5 (b) the affine space $\mathbb{A}^1$ has dimension 1, since the maximal chains of non-empty irreducible closed subsets of $\mathbb{A}^1$ are exactly $\{a\} \subsetneq \mathbb{A}^1$ for any point $a \in \mathbb{A}^1$. The codimension of $\{a\}$ in $\mathbb{A}^1$ is 1.

(b) One might be tempted to think that the "finiteness condition" of a Noetherian topological space X ensures that $\dim X$ is always finite. This is not true however: If we equip the natural numbers $X = \mathbb{N}$ with the topology in which (except $\emptyset$ and X) exactly the subsets $Y_n := \{0, \ldots, n\}$ for $n \in \mathbb{N}$ are closed, then X is Noetherian, but has chains $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n$ of non-empty irreducible closed subsets of arbitrary length.

However, for affine varieties infinite dimensions cannot occur, since in this case the notions of dimension and codimension reduce immediately to the concepts of the *Krull dimension* of a ring and the *codimension* (also called *height*) of a prime ideal that we know from commutative algebra [G3, Definition 11.1]:

Lemma 2.27 (Dimension and codimension of affine varieties)**.** *Let Y be a non-empty irreducible subvariety of an affine variety X.*

(a) *The dimension $\dim X$ of X is equal to the Krull dimension of the coordinate ring $A(X)$.*

(b) *The codimension $\operatorname{codim}_X Y$ of Y in X is equal to the codimension of the prime ideal $I(Y)$ in $A(X)$.*

In particular, dimensions and codimensions of (irreducible) affine varieties are always finite.

Proof. By Remark 2.9, chains of non-empty irreducible closed subsets of X (containing Y) correspond exactly to chains of prime ideals of $A(X)$ (contained in $I(Y)$). Hence, Definition 2.25 is directly equivalent to the definition of the Krull dimension of $A(X)$ and the codimension of $I(Y)$

in $A(X)$, respectively. By [G3, Remark 11.10], these numbers are finite since $A(X)$ is a finitely generated K-algebra. □

In fact, this correspondence allows us to transfer many results on Krull dimensions from commutative algebra immediately to statements on dimensions of affine varieties. For better reference, let us quickly recall the results that we will need in this class. We will list them only for irreducible varieties, since we will see in Remark 2.31 (a) that the general case can then easily be obtained by considering irreducible decompositions. The results are all very intuitive:

Proposition 2.28 (Properties of dimension)**.** *Let X and Y be non-empty irreducible affine varieties.*

(a) *We have* $\dim(X \times Y) = \dim X + \dim Y$ *(with $X \times Y$ having the Zariski topology as in Example 2.4 (c)). In particular,* $\dim \mathbb{A}^n = n$.

(b) *If $Y \subset X$ we have* $\dim X = \dim Y + \operatorname{codim}_X Y$. *In particular,* $\operatorname{codim}_X \{a\} = \dim X$ *for every point $a \in X$.*

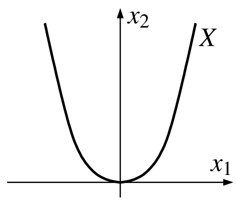
(c) *If $f \in A(X)$ is non-zero every irreducible component of $V(f)$ has codimension 1 in X (and hence dimension* $\dim X - 1$ *by (b)).*

Proof. Statement (a) is [G3, Proposition 11.9 (a) and Exercise 11.33 (a)]; the proof relies on Noether normalization.

Part (b) is [G3, Example 11.13 (a)]; the main non-trivial fact here (which would be false in arbitrary rings) is that all maximal chains of prime ideals in $A(X)$ have the same length, so that a maximal chain containing the prime ideal $I(Y)$ also has length $\dim X$.

Finally, statement (c) is just *Krull's Principal Ideal Theorem* [G3, Proposition 11.15 and Corollary
03 11.19]. □

Example 2.29. Let $X = V(x_2 - x_1^2) \subset \mathbb{A}^2_{\mathbb{C}}$ be the affine variety whose real points are shown in the picture on the right. Then we have as expected:



(a) X is irreducible by Proposition 2.8 since its coordinate ring $A(X) = \mathbb{C}[x_1,x_2]/(x_2 - x_1^2) \cong \mathbb{C}[x_1]$ is an integral domain.

(b) X has dimension 1 by Proposition 2.28 (c), since it is the zero locus of one non-zero polynomial in the affine space $\mathbb{A}^2$, and $\dim \mathbb{A}^2 = 2$ by Proposition 2.28 (a).

Exercise 2.30. Let A be an arbitrary subset of a topological space X. Prove that $\dim A \leq \dim X$.

Remark 2.31. Depending on where our chains of irreducible closed subvarieties end resp. start, we can break up the supremum in Definition 2.25 into irreducible components or local contributions:

(a) If $X = X_1 \cup \cdots \cup X_r$ is the irreducible decomposition of a Noetherian topological space as in Proposition 2.14, we have

$$\dim X = \max\{\dim X_1, \dots, \dim X_r\}:$$

"$\leq$" Assume that $\dim X \geq n$, so that there is a chain $Y_0 \subsetneq \cdots \subsetneq Y_n$ of non-empty irreducible closed subvarieties of X. Then $Y_n = (Y_n \cap X_1) \cup \cdots \cup (Y_n \cap X_r)$ is a union of closed subsets. So as Y_n is irreducible we must have $Y_n = Y_n \cap X_i$, and hence $Y_n \subset X_i$, for some i. But then $Y_0 \subsetneq \cdots \subsetneq Y_n$ is a chain of non-empty irreducible closed subsets in X_i, and hence $\dim X_i \geq n$.

"$\geq$" Let $\max\{\dim X_1, \dots, \dim X_r\} \geq n$. Then there is a chain of non-empty irreducible closed subsets $Y_0 \subsetneq \cdots \subsetneq Y_n$ in some X_i. This is also such a chain in X, and hence $\dim X \geq n$.

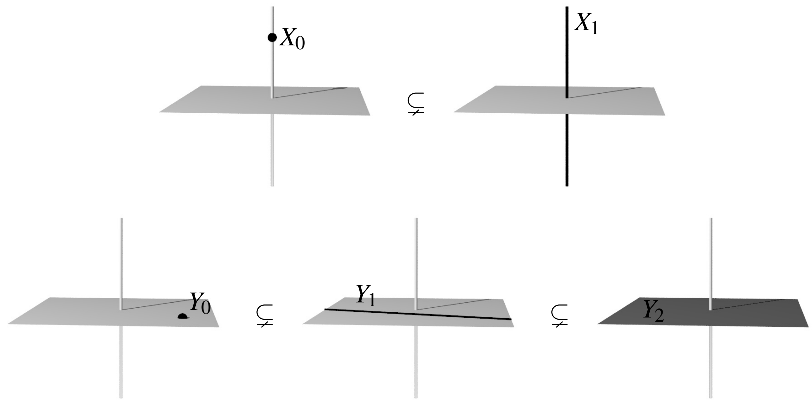
So for many purposes it suffices to consider the dimension of irreducible spaces.

(b) We always have $\dim X = \sup\{\operatorname{codim}_X \{a\} : a \in X\}$:

"$\leq$" If $\dim X \geq n$ there is a chain $Y_0 \subsetneq \cdots \subsetneq Y_n$ of non-empty irreducible closed subsets of X. For any $a \in Y_0$ this chain then shows that $\operatorname{codim}_X \{a\} \geq n$.

"$\geq$" If $\operatorname{codim}_X\{a\} \geq n$ for some $a \in X$ there is a chain $\{a\} \subset Y_0 \subsetneq \cdots \subsetneq Y_n$ of non-empty irreducible closed subsets of X, which also shows that $\dim X \geq n$.

The picture below illustrates these two equations: The affine variety $X = V(x_1x_3, x_2x_3) \subset \mathbb{A}^3$ is a union of two irreducible components, a line $V(x_1,x_2)$ of dimension 1 and a plane $V(x_3)$ of dimension 2 (see Proposition 2.28 (a)). So by (a) we have $\dim X = 2$ (with a maximal chain of length 2 as in Definition 2.25 (a) given by $Y_0 \subsetneq Y_1 \subsetneq Y_2$).



As for (b), the codimension of the point Y_0 is 2, whereas the codimension of the point X_0 is 1, as illustrated by the chains in the picture. Note that this codimension of a point can be interpreted geometrically as the *local dimension* of X at this point. Hence Proposition 2.28 (b) can also be interpreted as saying that the local dimension of an irreducible variety is the same at every point.

In practice, we will usually be concerned with affine varieties all of whose components have the same dimension. These spaces have special names that we want to introduce now. Note however that (as with the definition of a variety, see Remark 1.3) these terms are not used consistently throughout the literature — sometimes e. g. a curve is required to be irreducible, and sometimes it might be allowed to have additional components of dimension less than 1.

Definition 2.32 (Pure-dimensional spaces)**.**

(a) A Noetherian topological space X is said to be of **pure dimension** n if every irreducible component of X has dimension n.

(b) An affine variety is called ...

- a **curve** if it is of pure dimension 1;
- a **surface** if it is of pure dimension 2;
- a **hypersurface** in a pure-dimensional affine variety Y if it is an affine subvariety of Y of pure dimension $\dim Y - 1$.

Exercise 2.33. Let X be the set of all 2×3 matrices over a field K that have rank at most 1, considered as a subset of $\mathbb{A}^6 = \operatorname{Mat}(2 \times 3, K)$.

Show that X is an irreducible affine variety. What is its dimension?

Exercise 2.34. Let X be a topological space. Prove:

(a) If $\{U_i : i \in I\}$ is an open cover of X then $\dim X = \sup\{\dim U_i : i \in I\}$.

(b) If X is an irreducible affine variety and $U \subset X$ a non-empty open subset then $\dim X = \dim U$.

Does this statement hold more generally for any irreducible topological space?

Exercise 2.35. Let X be an affine variety with irreducible decomposition $X = X_1 \cup \cdots \cup X_r$, and let $a \in X$. Show that the local dimension $\operatorname{codim}_X\{a\}$ of X at a is given by

$$\operatorname{codim}_X\{a\} = \max\{\dim X_i : a \in X_i\}.$$

Exercise 2.36. Prove:

(a) Every Noetherian topological space is compact. In particular, every open subset of an affine variety is compact in the Zariski topology. (Recall that by definition a topological space X is compact if every open cover of X has a finite subcover.)

(b) A complex affine variety of dimension at least 1 is never compact in the classical topology.

We have seen in Proposition 2.28 (c) that the zero locus of a single polynomial in an irreducible affine variety is a hypersurface. Let us now address the opposite question: Is every (irreducible) hypersurface of a given irreducible affine variety X the zero locus of a single polynomial? Surprisingly, it turns out that the answer to this question depends on a rather subtle algebraic property of the coordinate ring $A(X)$ that will also be very important in the next chapter: the question whether it is a *unique factorization domain*, i. e. whether every non-zero non-unit in $A(X)$ can be written as a product of prime elements [G3, Definition 8.1]. The reason for this is the following result from commutative algebra.

Proposition 2.37. *Let R be a Noetherian integral domain (e. g. the coordinate ring $A(X)$ of an irreducible affine variety X). Then the following two statements are equivalent:*

(a) *Every prime ideal of codimension* 1 *in R is principal.*

(b) *R is a unique factorization domain.*

Proof.

(a) $\Rightarrow$ (b): First of all, similarly to the proof of Proposition 2.14 we can decompose any non-zero non-unit $f \in R$ as a product of irreducible elements since R is Noetherian: Otherwise f cannot be irreducible itself, so we must have a decomposition $f = f_1 f_1'$ into non-units, of which at least one factor, say f_1, is not a product of irreducible elements. We can then continue this process, i. e. write $f_1 = f_2 f_2'$ where f_2 is not a product of irreducibles, and so on. We thus obtain an infinite chain $\langle f \rangle \subsetneq \langle f_1 \rangle \subsetneq \langle f_2 \rangle \subsetneq \cdots$, in contradiction to R being Noetherian.

To prove that R is a unique factorization domain it therefore suffices to show that every irreducible element $f \in R$ is prime. To see this, choose a minimal prime ideal P containing f. By Krull's Principal Ideal Theorem [G3, Proposition 11.15] we then have $\operatorname{codim} P = 1$, so by assumption P is principal, i. e. we have $P = \langle g \rangle$ for a prime element g. But g divides f and f is irreducible, so that f and g agree up to units, and we obtain that f is prime as well.

(b) $\Rightarrow$ (a): Let P be a prime ideal of codimension 1 in R. We can then choose a non-zero element $f \in P$; since $P \neq \langle 1 \rangle$ it will also not be a unit.

As R is a unique factorization domain, we can thus write $f = f_1 \cdot \cdots \cdot f_k$ for some prime elements $f_1, \dots, f_k \in R$. Since P is a prime ideal we must then have $f_i \in P$ for some i. We thus obtain a chain $\{0\} \subsetneq \langle f_i \rangle \subset P$ of prime ideals in P. But as the codimension of P is 1 this requires that $P = \langle f_i \rangle$, i. e. that P is principal. □

Remark 2.38 (Ideal of hypersurfaces in $\mathbb{A}^n$)**.** Let X be an irreducible hypersurface in $\mathbb{A}^n$. Then $I(X) \trianglelefteq K[x_1, \dots, x_n]$ is a prime ideal of codimension 1. As the polynomial ring is a unique factorization domain [G3, Proposition 8.5] it follows from Proposition 2.37 that $I(X) = \langle f \rangle$ for an irreducible polynomial $f \in K[x_1, \dots, x_n]$.

If $X \subset \mathbb{A}^n$ is still a hypersurface, but not necessarily irreducible, we can apply the same argument to each component of its irreducible decomposition $X = X_1 \cup \cdots \cup X_k$ to obtain $I(X_j) = \langle f_j \rangle$ for some $f_j \in K[x_1, \dots, x_n]$ and all j. By Lemma 1.12 (a) the ideal $I(X) = \langle f \rangle$ with $f = f_1 \cdot \cdots \cdot f_k$ is then again principal.

As f is clearly unique up to units, we can associate its degree naturally to the hypersurface X:

Definition 2.39 (Degree of an affine hypersurface)**.** Let X be an affine hypersurface in $\mathbb{A}^n$, with ideal $I(X) = \langle f \rangle$ as in Remark 2.38. Then the degree of f is also called the **degree** of X, denoted $\deg X$. For the degrees 1, 2, 3 one calls X a **linear**, **quadric**, or **cubic** hypersurface, respectively.

In general, it is a hard problem to figure out if the coordinate ring of a given affine variety is a unique factorization domain. Let us therefore just give a single example in which this is not the case, and hence in which by Proposition 2.37 there is an irreducible codimension-1 hypersurface whose ideal is not principal.

Exercise 2.40. Let $R = K[x_1, x_2, x_3, x_4]/\langle x_1x_4 - x_2x_3 \rangle$. Show:

(a) R is an integral domain of dimension 3.

(b) $x_1, \dots, x_4$ are irreducible, but not prime in R. In particular, R is not a unique factorization domain.

(c) x_1x_4 and x_2x_3 are two decompositions of the same element of R into irreducible elements that do not agree up to units.

(d) $\langle x_1, x_2 \rangle$ is a prime ideal of codimension 1 in R that is not principal.

In particular, the plane $V(x_1, x_2)$ is a hypersurface in the affine variety $X = V(x_1x_4 - x_2x_3)$ whose ideal cannot be generated by one element in $A(X)$. 04