16. Cohomology of Sheaves

In this final chapter of these notes we want to give a short introduction to the important concept of cohomology of sheaves. There are two main motivations for this:

Remark 16.1 (Motivation for sheaf cohomology)**.**

(a) For a short exact sequence $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ of sheaves on a scheme X we have seen already in Exercise 13.26 (b) that taking global sections gives an exact sequence of $\mathscr{O}_X(X)$-modules (so in particular of Abelian groups)

$$0 \to \mathscr{F}_1(X) \to \mathscr{F}_2(X) \to \mathscr{F}_3(X).$$

However, Example 13.22 (b) showed us that the last map in this sequence is in general not surjective, so that we cannot obtain much information about $\mathscr{F}_3(X)$ from this. Cohomology gives a natural way to extend this sequence to the right: We will construct naturally defined cohomology groups $H^p(X,\mathscr{F})$ for any sheaf $\mathscr{F}$ on X and $p \in \mathbb{N}$ such that there is a long exact sequence

$$\begin{aligned} 0 &\to \mathscr{F}_1(X) \to \mathscr{F}_2(X) \to \mathscr{F}_3(X) \\ &\to H^1(X,\mathscr{F}_1) \to H^1(X,\mathscr{F}_2) \to H^1(X,\mathscr{F}_3) \\ &\to H^2(X,\mathscr{F}_1) \to \cdots . \end{aligned}$$

Quite often, there are explicit ways to compute these groups — and in fact they turn out to be zero in many cases, so that this a priori infinitely long exact sequence will usually break up into small and easily tractable pieces.

(b) If X is a variety over K, the cohomology groups $H^p(X,\mathscr{F})$ for a sheaf $\mathscr{F}$ on X will actually be K-vector spaces. Hence, applying this to canonically defined sheaves such as $\mathscr{O}_X$, Ω_X, or T_X (see Chapter 15), their dimensions are numbers attached intrinsically to X, so that they can be used to prove that varieties are not isomorphic.

The cohomology of sheaves is a very general concept. It can not only be defined for sheaves of modules on a scheme, but even for an arbitrary sheaf of Abelian groups on a topological space, and consequently it plays a big role in topology as well. Nevertheless, in order to avoid technicalities we will restrict to the case of quasi-coherent sheaves on varieties in these notes. In this case, the main idea for the construction of cohomology lies in Lemma 14.7 (b): If $X = \operatorname{Spec} R$ is an *affine* variety and $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ an exact sequence of quasi-coherent sheaves with $\mathscr{F}_i = \tilde{M}_i$ for R-modules M_i for $i \in \{1,2,3\}$, this means that the corresponding sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact, which by Proposition 14.2 (b) is just the sequence

$$0 \to \mathscr{F}_1(X) \to \mathscr{F}_2(X) \to \mathscr{F}_3(X) \to 0$$

of global sections. Hence, the problem of Remark 16.1 (a) of the sequence of global sections not being exact goes away in the affine case; it just arises from the gluing of affine patches. Consequently, in order to construct the cohomology of sheaves we will consider their sections on an affine open cover and study their gluing behavior. This is formalized by the following definition.

Definition 16.2. Let $\mathscr{F}$ be a sheaf on a variety X, and fix an affine open cover $U_1,\dots,U_r$ of X.

(a) For all $p \in \mathbb{N}$ we define the K-vector space

$$C^p(\mathscr{F}) = \bigoplus_{i_0<\cdots<i_p} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_p}).$$

Hence, an element $\varphi \in C^p(\mathscr{F})$ is just a collection of sections $\varphi_{i_0,\dots,i_p} \in \mathscr{F}(U_{i_0} \cap \dots \cap U_{i_p})$ for all intersections of $p+1$ sets taken from the chosen affine open cover. These sections can be completely unrelated.

(b) For every $p \in \mathbb{N}$ we define a linear so-called *boundary map* $d^p \colon C^p(\mathscr{F}) \to C^{p+1}(\mathscr{F})$ by

$$(d^p \varphi)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \varphi_{i_0,\dots,\hat{i_k},\dots,i_{p+1}}|_{U_{i_0}\cap\dots\cap U_{i_{p+1}}},$$

where the notation $\hat{i_k}$ means that the index i_k is to be left out. Note that this makes sense as $\varphi_{i_0,\dots,\hat{i_k},\dots,i_{p+1}}$ is a section of $\mathscr{F}$ on $U_{i_0} \cap \dots \cap \widehat{U_{i_k}} \cap \dots \cap U_{i_{p+1}}$, which contains $U_{i_0} \cap \dots \cap U_{i_{p+1}}$ as an open subset for all k.

Lemma 16.3. *For any sheaf $\mathscr{F}$ on a variety X, the composition $d^{p+1} \circ d^p \colon C^p(\mathscr{F}) \to C^{p+2}(\mathscr{F})$ is the zero map for all $p \in \mathbb{N}$.*

Proof. This statement holds because of the sign in the definition of the boundary map: Omitting for simplicity the restriction maps in the notation, we have for every $\varphi \in C^p(\mathscr{F})$

$$\begin{aligned}(d^{p+1}(d^p\varphi))_{i_0,\dots,i_{p+2}} &= \sum_{k=0}^{p+2} (-1)^k (d^p\varphi)_{i_0,\dots,\hat{i_k},\dots,i_{p+2}} \\ &= \sum_{k=0}^{p+2}\sum_{l=0}^{k-1} (-1)^{k+l} \varphi_{i_0,\dots,\hat{i_l},\dots,\hat{i_k},\dots,i_{p+2}} + \sum_{k=0}^{p+2}\sum_{l=k+1}^{p+2} (-1)^{k+l-1} \varphi_{i_0,\dots,\hat{i_k},\dots,\hat{i_l},\dots,i_{p+2}} \quad (*) \\ &= 0,\end{aligned}$$

where the exponent $l-1$ in the second sum of $(*)$ is due to the fact that the index i_l is at position $l-1$ as i_k has already been left out. Hence, the two sums in $(*)$ cancel each other since the second one is exactly the negative of the first. □

Remark 16.4 (Complexes of vector spaces)**.** Lemma 16.3 means that for any sheaf $\mathscr{F}$ on a variety X we have constructed a sequence of vector spaces and linear maps

$$C^0(\mathscr{F}) \xrightarrow{d^0} C^1(\mathscr{F}) \xrightarrow{d^1} C^2(\mathscr{F}) \xrightarrow{d^2} \cdots$$

such that the composition of any two subsequent maps is zero. This is usually called a *complex* of vector spaces. Note that this sequence is in general not exact, as we only have an inclusion $\operatorname{Im} d^p \subset \operatorname{Ker} d^{p+1}$ for all p, which might not be an equality. However, this inclusion allows us to form the quotient spaces $\operatorname{Ker} d^{p+1}/\operatorname{Im} d^p$ that measure "by how much the sequence fails to be exact" and are usually called the *cohomology* of this complex:

Definition 16.5 (Cohomology)**.** Let $\mathscr{F}$ be a sheaf on a variety X, and let $p \in \mathbb{N}$.

(a) With the notations of Definition 16.2, we define the p-th **cohomology** of $\mathscr{F}$ to be

$$H^p(X,\mathscr{F}) := \operatorname{Ker} d^p / \operatorname{Im} d^{p-1}$$

(with the convention that $C^{-1}(\mathscr{F})$ and d^{-1} are zero, so that $H^0(X,\mathscr{F}) = \operatorname{Ker} d^0$).

(b) The dimension of $H^p(X,\mathscr{F})$ as a K-vector space is denoted by $h^p(X,\mathscr{F})$.

Remark 16.6 (Independence of the affine cover)**.** As we have constructed it, we had to pick an affine open cover of a variety X in order to define the cohomology of a sheaf $\mathscr{F}$ on X. It is a crucial and non-trivial fact that up to isomorphism the resulting spaces $H^p(X,\mathscr{F})$ actually do not depend on this choice — as we have already indicated by the notation.

In fact, it is the main disadvantage of our construction that this independence is not obvious from the definition. There are other approaches to cohomology (e. g. the "derived functor approach" of [H, Chapter III]) that never use affine open covers and therefore do not have this problem, but that on the other hand are essentially useless for actual computations. We will therefore continue to use our construction above, assume the independence of the chosen affine cover for granted, and just refer to [H, Theorem III.4.5] for a proof.

We will start with a few simple cases in which the cohomology of sheaves can be obtained immediately from the definition.

Lemma 16.7 (First properties of cohomology)**.** *Let $\mathscr{F}$ be a sheaf on a variety X.*

(a) *We have $H^0(X,\mathscr{F}) \cong \mathscr{F}(X)$.*

(b) *If X is affine then $H^p(X,\mathscr{F}) = \{0\}$ for all $p > 0$.*

(c) *If X is projective of dimension n then $H^p(X,\mathscr{F}) = \{0\}$ for all $p > n$.*

(d) *If $i\colon X \to Y$ is the inclusion of X as a closed subvariety in Y then $H^p(Y,i_*\mathscr{F}) = H^p(X,\mathscr{F})$ for all $p \in \mathbb{N}$.*

Proof.

(a) By definition, we have $H^0(X,\mathscr{F}) = \operatorname{Ker}(d^0 \colon C^0(\mathscr{F}) \to C^1(\mathscr{F}))$. But an element $\varphi \in C^0(\mathscr{F})$ is given by sections $\varphi_i \in \mathscr{F}(U_i)$ for every affine open set U_i in the chosen cover, and the map d^0 is defined by

$$(d^0\varphi)_{i_0,i_1} = \varphi_{i_1}|_{U_{i_0}\cap U_{i_1}} - \varphi_{i_0}|_{U_{i_0}\cap U_{i_1}}$$

for all $i_0 < i_1$. By the sheaf axiom this is zero for all i_0 and i_1 if and only if the φ_i come from a global section of $\mathscr{F}$, so we conclude that $H^0(X,\mathscr{F}) \cong \mathscr{F}(X)$.

(b) Pick the affine open cover consisting of the single set X. Then $C^p(\mathscr{F}) = \{0\}$ for all $p > 0$ as there is no way to pick $p+1$ subsets from the cover, and hence $H^p(X,\mathscr{F}) = \{0\}$.

(c) By (the projective version of) Proposition 2.28 (c) we can recursively pick non-constant homogeneous polynomials $f_0,\dots,f_n$ such that $\dim(X \cap V(f_0,\dots,f_i)) < n-i$ for all $i = 0,\dots,n$. In particular, this means that $X \cap V(f_0,\dots,f_n) = \emptyset$, so that X is covered by the $n+1$ open complements of $V(f_i)$. But these open subsets are affine by Corollary 7.29, so we can choose them as the affine open cover for the construction of cohomology. As this cover has only $n+1$ subsets, we then have $C^p(\mathscr{F}) = \{0\}$ and hence $H^p(X,\mathscr{F}) = \{0\}$ for all $p > n$.

(d) Let $U_1,\dots,U_r$ be an affine open cover of Y. Then $X\cap U_1,\dots,X\cap U_r$ is an affine open cover of X, and with respect to these covers we have

$$C^p(i_*\mathscr{F}) = \bigoplus_{i_0<\cdots<i_p} i_*\mathscr{F}(U_{i_0}\cap\cdots\cap U_{i_p}) = \bigoplus_{i_0<\cdots<i_p} \mathscr{F}(X\cap U_{i_0}\cap\cdots\cap U_{i_p}) = C^p(\mathscr{F})$$

for all $p \in \mathbb{N}$. As the boundary maps for these two complexes agree as well, we conclude
27 that $H^p(Y,i_*\mathscr{F}) = H^p(X,\mathscr{F})$. □

As already mentioned in the introduction to this chapter, the most important tool for both computing and using cohomology is the long exact cohomology sequence arising from a short exact sequence of sheaves. It follows from a basic principle of commutative resp. homological algebra called the Snake Lemma.

Proposition 16.8 (Long exact cohomology sequence)**.** *Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be an exact sequence of sheaves on a variety X. Then there is a long exact sequence of vector spaces*

$$\begin{aligned} 0 &\to H^0(X,\mathscr{F}_1) \to H^0(X,\mathscr{F}_2) \to H^0(X,\mathscr{F}_3) \\ &\to H^1(X,\mathscr{F}_1) \to H^1(X,\mathscr{F}_2) \to H^1(X,\mathscr{F}_3) \\ &\to H^2(X,\mathscr{F}_1) \to \cdots. \end{aligned}$$

Proof. Consider the commutative diagram of linear maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^p(\mathscr{F}_1) & \longrightarrow & C^p(\mathscr{F}_2) & \longrightarrow & C^p(\mathscr{F}_3) & \longrightarrow & 0 \\ & & \big\downarrow d_1^p & & \big\downarrow d_2^p & & \big\downarrow d_3^p & & \\ 0 & \longrightarrow & C^{p+1}(\mathscr{F}_1) & \longrightarrow & C^{p+1}(\mathscr{F}_2) & \longrightarrow & C^{p+1}(\mathscr{F}_3) & \longrightarrow & 0, \end{array}$$

where the horizontal maps are induced by the given morphisms of sheaves. We claim that its rows are exact: By Exercise 5.23 (a), every intersection $U = U_{i_0}\cap\cdots\cap U_{i_p}$ of affine open subsets from the chosen cover is again an affine variety $\operatorname{Spec} R$. Hence, the three quasi-coherent sheaves $\mathscr{F}_i$ are associated to R-modules M_i for $i = 1,2,3$ on this intersection, and by Lemma 14.7 (b) the corresponding sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact. But by Proposition 14.2 (b) this is just the sequence of sections $0 \to \mathscr{F}_1(U) \to \mathscr{F}_2(U) \to \mathscr{F}_3(U) \to 0$. By Definition 16.2 the rows of the above diagram are just direct products of such sequences, and hence also exact.

The Snake Lemma [G3, Lemma 4.7] now implies that there is an induced exact sequence

$$0 \to \operatorname{Ker} d_1^p \to \operatorname{Ker} d_2^p \to \operatorname{Ker} d_3^p \to C^{p+1}(\mathscr{F}_1)/\operatorname{Im} d_1^p \to C^{p+1}(\mathscr{F}_2)/\operatorname{Im} d_2^p \to C^{p+1}(\mathscr{F}_3)/\operatorname{Im} d_3^p \to 0$$

made up from the kernels and cokernels of the vertical maps in the above diagram. Using the second half (for $p \to p-1$) and first half (for $p \to p+1$) of this sequence separately, we can make up a new commutative diagram with exact rows

$$\begin{array}{ccccccc} & C^p(\mathscr{F}_1)/\operatorname{Im} d_1^{p-1} & \longrightarrow & C^p(\mathscr{F}_2)/\operatorname{Im} d_2^{p-1} & \longrightarrow & C^p(\mathscr{F}_3)/\operatorname{Im} d_3^{p-1} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \operatorname{Ker} d_1^{p+1} & \longrightarrow & \operatorname{Ker} d_2^{p+1} & \longrightarrow & \operatorname{Ker} d_3^{p+1} & \end{array}$$

where the vertical maps are induced by d_i^p for $i = 1,2,3$. Applying the Snake Lemma again yields an exact sequence

$$H^p(X,\mathscr{F}_1) \to H^p(X,\mathscr{F}_2) \to H^p(X,\mathscr{F}_3) \to H^{p+1}(X,\mathscr{F}_1) \to H^{p+1}(X,\mathscr{F}_2) \to H^{p+1}(X,\mathscr{F}_3)$$

as the kernels and cokernels of the vertical maps in this new diagram are by definition just the cohomology spaces. The proposition is now obtained by combining these exact sequences for all p. □

The most important examples of sheaves are clearly the twisting sheaves on projective spaces, so let us explicitly compute their cohomology now. Most other sheaves that we have considered so far are related to them by exact sequences, so that their cohomology can then be computed as well by Proposition 16.8.

As the cohomology $H^p(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$ depends on three parameters p, n, and d, let us apply a notational trick to simplify the computations: We will determine the cohomology of the quasi-coherent (but not coherent) sheaf $\mathscr{F} := \oplus_{d\in\mathbb{Z}} \mathscr{O}_{\mathbb{P}^n}(d)$ on $\mathbb{P}^n$, whose sections are of the form $\frac{g}{f}$ with $f,g \in K[x_0,\dots,x_n]$ where f (but not necessarily g) is homogeneous. Note that $\mathscr{F}$ has a natural grading by d. Correspondingly, the following proposition computes its cohomology as graded vector spaces, so that we can recover from this all cohomology spaces $H^p(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$ by taking their d-th graded parts.

Proposition 16.9 (Cohomology of twisting sheaves on $\mathbb{P}^n$)**.** *Let $n \in \mathbb{N}_{>0}$, and consider the graded sheaf $\mathscr{F} := \bigoplus_{d\in\mathbb{Z}} \mathscr{O}_{\mathbb{P}^n}(d)$ on $\mathbb{P}^n$. Then, as graded vector spaces, we have*

(a) $H^0(\mathbb{P}^n,\mathscr{F}) \cong K[x_0,\dots,x_n]$;

(b) $H^n(\mathbb{P}^n,\mathscr{F}) \cong \frac{1}{x_0\cdot\dots\cdot x_n} K[x_0^{-1},\dots,x_n^{-1}]$;

(c) $H^p(\mathbb{P}^n,\mathscr{F}) = \{0\}$ *for all* $p \notin \{0,n\}$.

In particular, this means that

$$h^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{n} \text{ for } d \geq 0, \qquad h^n(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(d)) = \binom{-d-1}{n} \text{ for } d \leq -n-1,$$

and $h^p(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(d)) = 0$ *for all other p and d.*

Proof.

(a) As $H^0(\mathbb{P}^n,\mathscr{F})$ is just the space of global sections of $\mathscr{F}$ by Lemma 16.7 (a), this statement follows immediately from Example 13.5 (a).

(b) We compute this cohomology directly by definition, using the open cover by the affine spaces $U_i = \{x \in \mathbb{P}^n : x_i \neq 0\}$ for $i = 0,\dots,n$. Note that $\mathscr{F}(U_{i_0}\cap\dots\cap U_{i_p}) = K[x_0,\dots,x_n]_{x_{i_0}\cdot\dots\cdot x_{i_p}}$ is the vector space with basis the monomials $x_0^{k_0}\cdot\dots\cdot x_n^{k_n}$ where $k_{i_0},\dots,k_{i_p} \in \mathbb{Z}$ and all other exponents are non-negative. The relevant vector spaces for the computation of $H^n(\mathbb{P}^n,\mathscr{F})$ are thus

$$C^{n-1}(\mathscr{F})=\bigoplus_{i=0}^{n}K[x_0,\dots,x_n]_{x_0\cdot\cdots\cdot\hat{x}_i\cdot\cdots\cdot x_n}\quad\text{and}\quad C^n(\mathscr{F})=K[x_0,\dots,x_n]_{x_0\cdot\cdots\cdot x_n}$$

(together with $C^{n+1}(\mathscr{F})=\{0\}$), and hence we obtain

$$\begin{aligned}H^n(\mathbb{P}^n,\mathscr{F})&=C^n(\mathscr{F})/\operatorname{Im}(d^{n-1}\colon C^{n-1}(\mathscr{F})\to C^n(\mathscr{F}))\\&=\operatorname{Lin}\left(x_0^{k_0}\cdot\cdots\cdot x_n^{k_n}:\text{all }k_i\in\mathbb{Z}\right)/\operatorname{Lin}\left(x_0^{k_0}\cdot\cdots\cdot x_n^{k_n}:\text{at least one }k_i\geq 0\right)\\&=\operatorname{Lin}\left(x_0^{k_0}\cdot\cdots\cdot x_n^{k_n}:\text{all }k_i<0\right)\\&=\frac{1}{x_0\cdot\cdots\cdot x_n}K[x_0^{-1},\dots,x_n^{-1}].\end{aligned}$$

(c) We prove this statement by induction on n. As $H^p(\mathbb{P}^n,\mathscr{F})=\{0\}$ for $p>n$ by Lemma 16.7 (c) there is nothing to show for $n=1$. For $n>1$ we consider the exact ideal sequence of Lemma 14.8 for the inclusion $i\colon\mathbb{P}^{n-1}\cong V(x_0)\to\mathbb{P}^n$, together with Exercise 14.12:

$$0\longrightarrow\mathscr{O}_{\mathbb{P}^n}(-1)\xrightarrow{\cdot x_0}\mathscr{O}_{\mathbb{P}^n}\longrightarrow i_*\mathscr{O}_{\mathbb{P}^{n-1}}\longrightarrow 0.$$

Taking the tensor product with $\mathscr{O}_{\mathbb{P}^n}(d)$ keeps the sequence exact by Lemma 14.20 (a), and replaces $i_*\mathscr{O}_{\mathbb{P}^{n-1}}$ with

$$(i_*\mathscr{O}_{\mathbb{P}^{n-1}})\otimes\mathscr{O}_{\mathbb{P}^n}(d)\overset{14.15\,(b)}{\cong}i_*\big(\mathscr{O}_{\mathbb{P}^{n-1}}\otimes i^*\mathscr{O}_{\mathbb{P}^n}(d)\big)\cong i_*\big(\mathscr{O}_{\mathbb{P}^{n-1}}\otimes\mathscr{O}_{\mathbb{P}^{n-1}}(d)\big)\cong i_*\mathscr{O}_{\mathbb{P}^{n-1}}(d).$$

Hence, taking the direct sum over the resulting sequences for all $d\in\mathbb{Z}$ (which keeps the sequence exact as well), we obtain an exact sequence

$$0\longrightarrow\mathscr{F}\xrightarrow{\cdot x_0}\mathscr{F}\longrightarrow i_*\mathscr{F}\longrightarrow 0,$$

where $\mathscr{F}$ in the last term denotes the corresponding sheaf on $\mathbb{P}^{n-1}$. From the associated long exact cohomology sequence of Proposition 16.8 and the induction hypothesis we get the following exact sequences (using Lemma 16.7 (d) for the cohomology of $i_*\mathscr{F}$):

$$0\to H^0(\mathbb{P}^n,\mathscr{F})\overset{\cdot x_0}{\to}H^0(\mathbb{P}^n,\mathscr{F})\to H^0(\mathbb{P}^{n-1},\mathscr{F})\to H^1(\mathbb{P}^n,\mathscr{F})\overset{\cdot x_0}{\to}H^1(\mathbb{P}^n,\mathscr{F})\to 0,$$

$$0\to H^p(\mathbb{P}^n,\mathscr{F})\overset{\cdot x_0}{\to}H^p(\mathbb{P}^n,\mathscr{F})\to 0\quad\text{for }1<p<n-1,$$

$$0\to H^{n-1}(\mathbb{P}^n,\mathscr{F})\overset{\cdot x_0}{\to}H^{n-1}(\mathbb{P}^n,\mathscr{F})\to H^{n-1}(\mathbb{P}^{n-1},\mathscr{F})\to H^n(\mathbb{P}^n,\mathscr{F})\overset{\cdot x_0}{\to}H^n(\mathbb{P}^n,\mathscr{F})\to 0.$$

The second sequence tells us that multiplication with x_0 is an isomorphism from $H^p(\mathbb{P}^n,\mathscr{F})$ to itself for $1<p<n-1$. But the same is true for $p=1$ as the first sequence starts by (a) with

$$0\to K[x_0,\dots,x_n]\overset{\cdot x_0}{\to}K[x_0,\dots,x_n]\to K[x_1,\dots,x_n]\to\cdots,$$

which is obviously also exact on the right. The same analysis for the third sequence using (b) shows that multiplication with x_0 is in fact an isomorphism from $H^p(\mathbb{P}^n,\mathscr{F})$ to itself for all $1\leq p\leq n-1$.

Moreover, this multiplication with x_0 makes $H^p(\mathbb{P}^n,\mathscr{F})$ into a $K[x_0]$-module. As localization is exact, localizing this module at x_0 can be performed by taking the cohomology of the complex of the localized modules $C^p(\mathscr{F})_{x_0}=C^p(\mathscr{F}|_{U_0})$. But this cohomology vanishes for $p>0$ by Lemma 16.7 (b) since $U_0\cong\mathbb{A}^n$ is affine. Hence we obtain

$$H^p(\mathbb{P}^n,\mathscr{F})_{x_0}\cong H^p(U_0,\mathscr{F}|_{U_0})=\{0\}.$$

So for any $\varphi\in H^p(\mathbb{P}^n,\mathscr{F})$ it follows that $x_0^k\cdot\varphi=0$ for some k. But we have shown above that multiplication with x_0 in $H^p(\mathbb{P}^n,\mathscr{F})$ is an isomorphism, hence $\varphi=0$. This means that $H^p(\mathbb{P}^n,\mathscr{F})=0$ as desired. □

Example 16.10 (The double skyscraper sequence revisited)**.** Recall from Example 13.22 (b) the double skyscraper sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(-2) \stackrel{\cdot x_0 x_1}{\longrightarrow} \mathscr{O}_{\mathbb{P}^1} \stackrel{f}{\longrightarrow} K_P \oplus K_Q \longrightarrow 0$$

on $\mathbb{P}^1$, where $P = (1:0)$ and $Q = (0:1)$, and f is the evaluation of a regular function at P and Q. We have seen already that this sequence is exact, but that f is not surjective on global sections. As global sections are the same as the 0-th cohomology by Lemma 16.7 (a), Proposition 16.8 now continues the sequence of global sections to the right, with the first terms of the resulting long exact cohomology sequence being

$$0 \to H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2)) \to H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) \to H^0(\mathbb{P}^1, K_P \oplus K_Q) \to H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2)) \to H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}).$$

But we know all these terms already: By Proposition 16.9 (a) we have $h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2)) = 0$ and $h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = 1$, and by Proposition 16.9 (b) we have $h^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2)) = 1$ and $h^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = 0$. Moreover, $H^0(\mathbb{P}^1, K_P \oplus K_Q)$ is isomorphic to $K \oplus K$ given by specifying values at the points P and Q. Hence, skipping the first redundant 0, the above exact cohomology sequence becomes

$$0 \to K \to K \oplus K \to K \to 0.$$

To get a better understanding of this exact sequence, we can actually also identify the morphisms in it:

- The first non-trivial morphism is just

  $$K \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) \to H^0(\mathbb{P}^1, K_P \oplus K_Q) \cong K \oplus K, \; a \mapsto (a, a)$$

  as it is the evaluation of the constant function a on $\mathbb{P}^1$ at the points P and Q.
- The second morphism $K \oplus K \cong H^0(\mathbb{P}^1, K_P \oplus K_Q) \to H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2)) \cong K$ is the connecting homomorphism in the Snake Lemma for the diagram

  $$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0(\mathscr{O}_{\mathbb{P}^1}(-2)) & \longrightarrow & C^0(\mathscr{O}_{\mathbb{P}^1}) & \longrightarrow & C^0(K_P \oplus K_Q) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow d^0 & & \downarrow & & \\ 0 & \longrightarrow & C^1(\mathscr{O}_{\mathbb{P}^1}(-2)) & \longrightarrow & C^1(\mathscr{O}_{\mathbb{P}^1}) & \longrightarrow & C^1(K_P \oplus K_Q) & \longrightarrow & 0 \end{array}$$

  that we have already considered in the proof of Proposition 16.8. Hence, starting with any element $(a, b) \in C^0(K_P \oplus K_Q)$ representing a cohomology class in $H^0(\mathbb{P}^1, K_P \oplus K_Q)$ we can firstly find an inverse image in $C^0(\mathscr{O}_{\mathbb{P}^1}) = \mathscr{O}_{\mathbb{P}^1}(U_0) \oplus \mathscr{O}_{\mathbb{P}^1}(U_1)$ (with $U_i = \{x \in \mathbb{P}^1 : x_i \neq 0\}$), namely the pair of constant functions (a, b) (as $P \in U_0$ and $Q \in U_1$). Going down in the above diagram yields the function $b - a \in C^1(\mathscr{O}_{\mathbb{P}^1}) = \mathscr{O}_{\mathbb{P}^1}(U_0 \cap U_1)$ by the definition of the boundary map. Finally, as the morphism from $\mathscr{O}_{\mathbb{P}^1}(-2)$ to $\mathscr{O}_{\mathbb{P}^1}$ is given by multiplication with $x_0 x_1$, we find that $\frac{b-a}{x_0 x_1}$ is the element in $C^1(\mathscr{O}_{\mathbb{P}^1}(-2)) = \mathscr{O}_{\mathbb{P}^1}(2)(U_0 \cap U_1)$ that we were looking for. Note that $\frac{1}{x_0 x_1}$ is in fact a basis vector of $H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2))$ by Proposition 16.9 (b).

Hence, in this basis our cohomology sequence reads

$$\begin{array}{ccccccccc} 0 & \to & K & \to & K \oplus K & \to & K & \to & 0 \\ & & a & \mapsto & (a, a) & & & & \\ & & & & (a, b) & \mapsto & b - a, & & \end{array}$$

which is immediately seen to be exact.

Let us now apply the theory of cohomology of sheaves to the classification of varieties, i. e. to prove that varieties are not isomorphic. As already mentioned in Remark 16.1 (b), the idea is to consider dimensions of cohomology spaces for canonically defined sheaves such as the structure sheaf or the cotangent sheaf, as these numbers would have to be the same for isomorphic varieties. To simplify computations, it is often useful however not to consider these dimensions separately but a certain linear combination of them instead:

Construction 16.11 (Euler characteristic)**.** Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be an exact sequence of sheaves on a projective variety X. We have seen in Proposition 16.8 that there is then a long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(X,\mathscr{F}_1) \to H^0(X,\mathscr{F}_2) \to H^0(X,\mathscr{F}_3) \\ &\to H^1(X,\mathscr{F}_1) \to H^1(X,\mathscr{F}_2) \to H^1(X,\mathscr{F}_3) \\ &\to H^2(X,\mathscr{F}_1) \to \cdots . \end{aligned} \qquad (*)$$

As we have assumed X to be projective, note that this is in fact a finite sequence by Lemma 16.7 (c). Moreover, if we knew that all $H^p(X,\mathscr{F}_i)$ are finite-dimensional vector spaces, it is a standard commutative algebra result [G3, Corollary 4.10] that the exactness of this sequence implies

$$h^0(X,\mathscr{F}_1) - h^0(X,\mathscr{F}_2) + h^0(X,\mathscr{F}_3) - h^1(X,\mathscr{F}_1) + h^1(X,\mathscr{F}_2) - h^1(X,\mathscr{F}_3) \pm \cdots = 0.$$

So if for any sheaf $\mathscr{F}$ on X we set

$$\chi(X,\mathscr{F}) := \sum_{p=0}^{\infty} (-1)^p \, h^p(X,\mathscr{F})$$

(which again is actually a finite sum) it follows that

$$\chi(X,\mathscr{F}_2) = \chi(X,\mathscr{F}_1) + \chi(X,\mathscr{F}_3).$$

The number $\chi(X,\mathscr{F})$ is called the **Euler characteristic** of $\mathscr{F}$. The fact that it is "additive on exact sequences" in the sense above makes it usually very easy to compute for a given sheaf, and thus we will see below that it often provides a good way to show that varieties are not isomorphic.

As for the finite-dimensionality of the cohomology, note that an exact sequence $(*)$ also implies that all vector spaces $H^p(X,\mathscr{F}_i)$ of a sheaf $\mathscr{F}_i$ for $i \in \{1,2,3\}$ are finite-dimensional if this is true for the two others. But all our examples below will be built up from exact sequences in this way starting from the twisting sheaves on projective spaces — for which we have shown their finite-dimensionality already in Proposition 16.9. Actually, one can even show that every coherent sheaf on a projective variety has finite-dimensional cohomology [H, Theorem III.5.2], but we will neither need nor prove this here.

Example 16.12 (Euler characteristic of twisting sheaves on $\mathbb{P}^n$)**.** If we define the binomial coefficient

$$\binom{a}{n} = \frac{a \cdot (a-1) \cdot \cdots \cdot (a-n+1)}{n!}$$

for all $a \in \mathbb{Z}$ and $n \in \mathbb{N}$ we have

$$\chi(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{n}$$

for all $n \in \mathbb{N}_{>0}$ and $d \in \mathbb{Z}$ by Proposition 16.9:

- for $d \geq 0$ we have $\chi(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = h^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{n}$;
- for $d \leq -n-1$ we have $\chi(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = (-1)^n \, h^n(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = (-1)^n \binom{-d-1}{n} = \binom{n+d}{n}$;
- for $-n-1 < d < 0$ we have $\chi(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = 0 = \binom{n+d}{n}$.

Corollary 16.13 (Euler characteristic of twisting sheaves on hypersurfaces)**.** *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d, and let $e \in \mathbb{Z}$. Then*

$$\chi(X, \mathscr{O}_X(e)) = \binom{n+e}{n} - \binom{n+e-d}{n}.$$

Proof. By Lemma 14.8 (b) and Exercise 14.12 we have the exact ideal sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-d) \to \mathscr{O}_{\mathbb{P}^n} \to i_* \mathscr{O}_X \to 0$$

on $\mathbb{P}^n$, where $i\colon X \to \mathbb{P}^n$ denotes the inclusion map. The sequence remains exact by Lemma 14.20 (a) if we take the tensor product with the line bundle $\mathscr{O}_{\mathbb{P}^n}(e)$. As $i_*\mathscr{O}_X \otimes \mathscr{O}_{\mathbb{P}^n}(e) \cong i_*\mathscr{O}_X(e)$ by the projection formula of Lemma 14.15 (b), we thus obtain the new exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n}(e-d) \to \mathscr{O}_{\mathbb{P}^n}(e) \to i_*\mathscr{O}_X(e) \to 0.$$

Using the additivity of the Euler characteristic of Construction 16.11 then yields

$$\begin{aligned}\chi(X,\mathscr{O}_X(e)) &\overset{16.7\,(d)}{=} \chi(\mathbb{P}^n, i_*\mathscr{O}_X(e)) = \chi(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(e)) - \chi(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(e-d)) \\ &\overset{16.12}{=} \binom{n+e}{n} - \binom{n+e-d}{n},\end{aligned}$$

as we have claimed. □

Example 16.14.

(a) (Genus of a plane curve) By Corollary 16.13, the Euler characteristic of the structure sheaf of a plane curve $X \subset \mathbb{P}^2$ of degree d is

$$h^0(X,\mathscr{O}_X) - h^1(X,\mathscr{O}_X) = \chi(X,\mathscr{O}_X) = 1 - \binom{2-d}{2} = \frac{-d^2+3d}{2}.$$

But X must be connected by Exercise 6.31, so we have $h^0(X,\mathscr{O}_X) = 1$, and thus

$$h^1(X,\mathscr{O}_X) = \frac{(d-1)(d-2)}{2}. \qquad (*)$$

This number $h^1(X,\mathscr{O}_X)$ (in fact, for any projective curve X) is called the **genus** of X; it is clearly invariant under isomorphisms. Note that this coincides with the definition of genus in the "Plane Algebraic Curves" class [G2, Definition 8.10], where we have also seen that over the ground field $\mathbb{C}$ the genus g can be interpreted topologically as X is then a "sphere with g handles" [G2, Remark 5.12], e. g. a sphere for $g = 0$ as in Example 13.1 and a torus for $g = 1$.

As the genus in $(*)$ is a different number for all $d > 1$, it follows now that plane curves of different degrees bigger than 1 can never be isomorphic. In contrast, a line (i. e. a plane curve of degree 1) and a conic (irreducible of degree 2) are always isomorphic by Example 7.5 (d), and consequently their genus 0 is the same.

(b) Note that Euler characteristics of twisting sheaves of a plane curve X of degree d such as

$$\chi(X,\mathscr{O}_X(1)) = \binom{3}{2} - \binom{3-d}{2} = \frac{-d^2+5d}{2}$$

are *not* invariant under isomorphisms since the twisting sheaves depend on the projective embedding (which need not be preserved under isomorphisms). For example, by the above formula $\chi(X,\mathscr{O}_X(1))$ is equal to 2 for a line and 3 for a conic although these curves are isomorphic.

(c) For a surface $X \subset \mathbb{P}^3$ of degree d we have by Corollary 16.13

$$\chi(X,\mathscr{O}_X) = 1 - \binom{3-d}{3},$$

and of course this number is again invariant under isomorphisms. But note that it is equal to 1 for all $d \in \{1,2,3\}$ although these three cases look very different: For $d = 1$ we obtain $\mathbb{P}^2$, the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in Example 7.11 yields a surface of degree $d = 2$, and for $d = 3$ cubic surfaces in Chapter 11 had again very different properties. To prove rigorously that surfaces of degrees 1, 2, and 3 indeed cannot be isomorphic we need a different invariant, e. g. one obtained from the cotangent sheaf:

Corollary 16.15 (Euler characteristic of the cotangent sheaf for hypersurfaces)**.** *For any hypersurface X of degree d in projective space $\mathbb{P}^n$ over a field of characteristic* 0 *we have*

$$\chi(X,\Omega_X) = \binom{n-2d}{n} - (n+1)\binom{n-d-1}{n} - 1.$$

Proof. The Euler sequence of Proposition 15.8 is a sequence of vector bundles, so by Lemma 14.20 (b) we can pull it back along the inclusion $i\colon X \to \mathbb{P}^n$ to obtain the exact sequence

$$0 \to i^*\Omega_{\mathbb{P}^n} \to \mathscr{O}_X(-1)^{n+1} \to \mathscr{O}_X \to 0$$

on X. Moreover, there is the exact conormal sequence of Proposition 15.10

$$0 \to \mathscr{O}_X(-d) \to i^*\Omega_{\mathbb{P}^n} \to \Omega_X \to 0.$$

Combining these two sequences, we get by the additivity of the Euler characteristic

$$\begin{aligned}
\chi(X,\Omega_X) &= \chi(X,i^*\Omega_{\mathbb{P}^n}) - \chi(X,\mathscr{O}_X(-d)) \\
&= (n+1)\,\chi(X,\mathscr{O}_X(-1)) - \chi(X,\mathscr{O}_X) - \chi(X,\mathscr{O}_X(-d)) \\
&\overset{16.13}{=} \binom{n-2d}{n} - (n+1)\binom{n-d-1}{n} - 1.
\end{aligned}$$ □

Example 16.16.

(a) For a plane curve $X \subset \mathbb{P}^2$ of degree d we have

$$\chi(X,\Omega_X) = \frac{d^2-3d}{2}$$

by Corollary 16.15. This is again a number invariant under isomorphisms, but note that it is just the negative of the invariant $\chi(X,\mathscr{O}_X)$ that we have already found in Example 16.14 (a).

In fact, this is not a coincidence: There is a deep result in algebraic geometry called *Serre duality* [H, Chapter III.7] implying for any coherent sheaf $\mathscr{F}$ on a smooth projective curve X that there are natural isomorphisms

$$H^p(X,\mathscr{F})^\vee \cong H^{1-p}(X,\Omega_X \otimes \mathscr{F}^\vee) \quad \text{for } p \in \{0,1\}.$$

Applying this for $\mathscr{F} = \mathscr{O}_X$ we see that

$$\chi(X,\Omega_X) = h^0(X,\Omega_X) - h^1(X,\Omega_X) = h^1(X,\mathscr{O}_X) - h^0(X,\mathscr{O}_X) = -\chi(X,\mathscr{O}_X).$$

In fact, there is also a version of Serre duality on higher-dimensional varieties, which explains e. g. the symmetry between the parts (a) and (b) of Proposition 16.9.

(b) If $X \subset \mathbb{P}^3$ is a surface of degree d then Corollary 16.15 shows that

$$\chi(X,\Omega_X) = \frac{-2d^3+6d^2-7d}{3}.$$

It is easy to see that these are all different numbers for all $d \in \mathbb{N}_{>0}$, so we conclude that
28 surfaces in $\mathbb{P}^3$ of different degrees can never be isomorphic.