15. Differentials

We have seen already in Proposition 10.11 that (formal) differentiation of functions is useful to compute the tangent spaces at the (closed) points of a variety X. We now want to introduce this language of differentials in general. The idea behind this is that the various tangent *spaces* $T_P X$ for $P \in X$ should not just be independent vector spaces at every point, but rather arise as the fibers of one globally defined tangent *sheaf* on X, as already motivated in Example 13.1.

To construct this tangent sheaf rigorously, we will restrict to the case of varieties as we have defined the notions of tangent spaces and smoothness only in this case. In particular, there will always be a fixed algebraically closed ground field K in this chapter. As the tangent sheaf that we are going to construct will be quasi-coherent, let us first define a suitable module over the coordinate ring R of an affine variety. The following notion of differentials captures the formal properties that we would expect from the differentiation of functions in this case.

Definition 15.1 (Differentials)**.** Let R be a K-algebra. We define Ω_R to be the free R-module generated by formal symbols df for all $f \in R$, modulo the relations

- $d(f+g) = df + dg$ for all $f, g \in R$;
- $d(fg) = f\,dg + g\,df$ for all $f, g \in R$;
- $df = 0$ for all $f \in K$.

The elements of Ω_R are called **(Kähler) differentials** of R (over K).

We can thus consider d as a map that sends an element $f \in R$ to its differential $df \in \Omega_R$. Note however that, because of the rule for the differentiation of products, this map $d\colon R \to \Omega_R$ is only K-linear but not an R-module homomorphism — although both R and Ω_R are R-modules.

Remark 15.2 (Differentials and localization)**.** Let S be a multiplicatively closed subset in a K-algebra R. Then for any $g \in R$ and $f \in S$ we have in $\Omega_{S^{-1}R}$

$$0 = d\left(\frac{1}{f} \cdot f\right) = \frac{1}{f}\,df + f\,d\frac{1}{f}, \quad \text{hence} \quad d\left(\frac{1}{f}\right) = -\frac{1}{f^2}\,df, \quad \text{and thus} \quad d\left(\frac{g}{f}\right) = \frac{1}{f}\,dg - \frac{g}{f^2}\,df.$$

In other words, the differentials of the localized algebra satisfy not only the standard rules of differentiation for sums, products, and constants, but also for quotients.

This computation also shows that the differential of a quotient $\frac{g}{f} \in S^{-1}R$ can be expressed as an $S^{-1}R$-linear combination of the differentials of $f, g \in R$. In this way we get an isomorphism of $S^{-1}R$-modules

$$\Omega_{S^{-1}R} \to S^{-1}\Omega_R,\ d\left(\frac{g}{f}\right) \mapsto \frac{1}{f}\,dg - \frac{g}{f^2}\,df$$

(in fact, it is easy to check that this is well-defined and bijective), i. e. modules of differentials commute with localization.

In particular, it follows that on an affine variety $\operatorname{Spec} R$ there is also a well-defined notion of differentiation of regular functions (given by elements of R_P for all $P \in \operatorname{Spec} R$ by Definition 12.16) that yields sections of the quasi-coherent sheaf $\tilde{\Omega}_R$ (given by elements of $(\Omega_R)_P \cong \Omega_{R_P}$ by Definition 14.1). Hence d extends to a map of sheaves $d\colon \mathscr{O}_{\operatorname{Spec} R} \to \tilde{\Omega}_R$; but note again that this is *not* a morphism of sheaves of modules as it does not commute with products.

In the following, we will often also use the differentiation operator d in this extended version without further notice.

Example 15.3.

(a) Let $R = K[x_1,\dots,x_n]$ be the polynomial ring. By the rules of differentiation imposed in Definition 15.1 we have $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\,dx_i$ for all $f \in R$, so Ω_R is generated by $dx_1,\dots,dx_n$ as an R-module. Moreover, there are no further relations among these differentials, so

$$\Omega_R = R\,dx_1 \oplus \cdots \oplus R\,dx_n$$

is in fact a free R-module of rank n, with basis the differentials of the coordinates.

(b) More generally, consider the coordinate ring $R = A(X) = K[x_1,\dots,x_n]/I(X)$ of an affine variety $X \subset \mathbb{A}^n$. As in (a), Ω_R is then still generated by $dx_1,\dots,dx_n$ as an R-module, but in addition for all $f \in I(X)$ we have $f = 0$ in R, and hence also $df = 0$. It suffices to impose these conditions for generators of $I(X)$, and thus we obtain

$$\Omega_R = R\,dx_1 \oplus \cdots \oplus R\,dx_n \Big/ \left\langle \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}\,dx_j : i = 1,\dots,m \right\rangle \quad \text{for} \quad I(X) = \langle f_1,\dots,f_m \rangle.$$

In particular, for a (closed) point $P \in X$, i. e. a maximal ideal $P \trianglelefteq R$ (so that $R/P \cong K$), we have

$$\Omega_R \otimes_R R/P = K\,dx_1 \oplus \cdots \oplus K\,dx_n \Big/ \left\langle \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(P)\,dx_j : i = 1,\dots,m \right\rangle,$$

which by (the proof of) the Jacobi criterion in Proposition 10.11 is just the dual $(T_P X)^\vee$ of the tangent space $T_P X$. As motivated in the introduction to this chapter, this means in the language of Remark 14.14 (b) that we have constructed a quasi-coherent sheaf $\tilde{\Omega}_R$ on X whose fiber at every point P is precisely the dual of the tangent space $T_P X$.

We now have to globalize this construction to a quasi-coherent sheaf on an arbitrary variety. Unfortunately, Definition 15.1 does not glue very well, so we have to give an alternative description of differentials first. Similarly to the definition of the pull-back of sheaves in Construction 14.13 (b), its only purpose for us is to show the existence of a sheaf of differentials in the general case; for actual (local) computations we will always use the module Ω_R from above.

Lemma 15.4 (Alternative description of Ω_R)**.** *Let R be a K-algebra. We consider the map*

$$\delta\colon R \otimes_K R \to R,\ f \otimes g \mapsto fg$$

and set $J := \operatorname{Ker}\delta$. Then J/J^2 is an R-module isomorphic to Ω_R.

Proof. Note first that $R \otimes_K R$ is an R-algebra in two ways, by multiplication in the left and in the right factor. For both choices, J^2 is well-defined as the R-submodule of J generated by all products $f_1 f_2 \otimes g_1 g_2$ with $f_1 \otimes g_1, f_2 \otimes g_2 \in J$. But in fact, in the quotient J/J^2 both R-module structures coincide, since for all $h \in R$ and $\sum_{i=1}^n f_i \otimes g_i \in J$ we have

$$\sum_{i=1}^n (f_i \otimes hg_i - hf_i \otimes g_i) = \sum_{i=1}^n (f_i \otimes g_i) \cdot \underbrace{(1 \otimes h - h \otimes 1)}_{\in J} \quad \in J^2.$$

For this R-module structure of J/J^2 it is now straightforward to check that the maps

$$J/J^2 \to \Omega_R,\ \sum_{i=1}^n f_i \otimes g_i \mapsto f_i\,dg_i$$

$$\text{and} \quad \Omega_R \to J/J^2,\ df \mapsto 1 \otimes f - f \otimes 1$$

are well-defined R-module homomorphisms and inverse to each other. □

Construction 15.5 (Cotangent sheaf)**.** Let X be a variety. By Definition 5.17, the diagonal Δ_X is then a closed subvariety of $X \times X$ isomorphic to X. We denote by $i\colon X \cong \Delta_X \to X \times X$ the inclusion, and by $\mathscr{I} := \mathscr{I}_{X/X\times X}$ its ideal sheaf on $X \times X$ as in Lemma 14.8.

Note that in the affine case $X = \operatorname{Spec} R$ the inclusion i corresponds to the ring homomorphism δ of Lemma 15.4, the ideal sheaf $\mathscr{I}$ is the sheaf associated to its kernel J, and pulling back $\mathscr{I}/\mathscr{I}^2$ by the map i considers J/J^2 as an R-module instead of as an $(R \otimes_K R)$-module. Hence, for a general variety X we define the **cotangent sheaf** of X as

$$\Omega_X := i^*(\mathscr{I}/\mathscr{I}^2).$$

By construction, Lemma 15.4 then means that Ω_X restricts to the sheaf $\tilde{\Omega}_R$ on an affine open subset $\operatorname{Spec} R$ of X. As in Remark 15.2, there is still a map of sheaves $d\colon \mathscr{O}_X \to \Omega_X$ (which is not a morphism of sheaves of modules) that restricts to taking differentials on affine open subsets.

If X is a smooth variety of pure dimension n we know by Lemma 10.9 that all tangent spaces $T_P X$ for $P \in X$ (and hence also all cotangent spaces $(T_P X)^\vee$) have dimension n. Hence, we would expect Ω_X to be a vector bundle of rank n in this case. Let us prove this now, so that we can then define the tangent bundle as its dual bundle.

Proposition 15.6. *Let X be a variety of pure dimension n. Then Ω_X is locally free of rank n if and only if X is smooth.*

Proof.

"⇒" If Ω_X is a vector bundle of rank n then its fiber at any point $P \in X$, i. e. by Example 15.3 (b) the cotangent space $(T_P X)^\vee$, has dimension n. Hence $T_P X$ has dimension n as well, which means by Lemma 10.9 that P is a smooth point of X.

"⇐" Now let us assume that X is smooth, and let $P \in X$. We may assume that $X \subset \mathbb{A}^r$ is affine, with coordinate ring $R = A(X) = K[x_1, \dots, x_r]/\langle f_1, \dots, f_m \rangle$. As in Example 15.3 (b) we then have

$$(T_P X)^\vee = K\,dx_1 \oplus \cdots \oplus K\,dx_r \Big/ \left\langle \sum_{j=1}^{r} \frac{\partial f_i}{\partial x_j}(P)\,dx_j : i = 1, \dots, m \right\rangle.$$

As this vector space has dimension n by assumption, the Jacobian matrix $J(P) = \left(\frac{\partial f_i}{\partial x_j}(P)\right)_{i,j}$ at P has rank $r-n$. So without loss of generality we may assume that the submatrix of $J(P)$ given by the last $r-n$ rows and columns has a non-zero determinant. This means that the differentials $dx_{n+1}, \dots, dx_r$ in $(T_P X)^\vee$ can be expressed as a K-linear combination of $dx_1, \dots, dx_n$ (and that $dx_1, \dots, dx_n$ are a basis of $(T_P X)^\vee$).

Now let $h \in R$ be the determinant of the last $r-n$ rows and columns of the Jacobian matrix $J = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$. Restricting to the smaller open subset $D(h) = \operatorname{Spec} R_h$ (which contains P since $h(P) \neq 0$) we can then invert this determinant, and hence express $dx_{n+1}, \dots, dx_r$ in Ω_{R_h} as an R_h-linear combination of $dx_1, \dots, dx_n$. This means that $dx_1, \dots, dx_n$ even suffice to generate Ω_{R_h} as an R_h-module.

Moreover, there are no relations among $dx_1, \dots, dx_n$ in Ω_{R_h}: If we had a non-trivial relation $g_1\,dx_1 + \cdots + g_n\,dx_n = 0$ with $g_1, \dots, g_n \in R_h$, without loss of generality with $g_1 \neq 0$, there would be a point $Q \in D(h)$ with $g_1(Q) \neq 0$ by the Nullstellensatz, and hence a non-trivial relation $g_1(Q)\,dx_1 + \cdots + g_n(Q)\,dx_n = 0$ in $(T_Q X)^\vee$. But this is impossible since we then had $\dim (T_Q X)^\vee < n$, in contradiction to Remark 10.2 (c).

Hence, we see that Ω_{R_h} is a free R_h-module of rank n with generators $dx_1, \dots, dx_n$, i. e. that Ω_X is locally free of rank n. □

Definition 15.7 (Tangent bundle)**.** For a smooth variety X of pure dimension n, the cotangent sheaf Ω_X is often called the **cotangent bundle** as it is a vector bundle of rank n by Proposition 15.6. Its dual $T_X := \Omega_X^\vee$, which is then also a vector bundle of rank n, is called the **tangent sheaf** or **tangent bundle** of X.

The importance of the cotangent resp. tangent bundle stems from the fact that they are *canonically defined* for any smooth variety. This gives e. g. powerful methods to show that two varieties are not isomorphic: If, for example, we have two varieties whose cotangent bundles have different properties (say their spaces of global sections have different dimensions), then these varieties cannot be isomorphic. We will explore this idea later in Example 16.16, but first let us see how these bundles can actually be computed in some of the most important cases, namely for projective spaces and their hypersurfaces. In both these cases they are determined by exact sequences in terms of other bundles that we already know.

Proposition 15.8 (Euler sequence). *For all $n \in \mathbb{N}_{>0}$ the cotangent bundle of $\mathbb{P}^n$ is determined by the exact sequence*

$$0 \to \Omega_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathscr{O}_{\mathbb{P}^n} \to 0,$$

where $\mathscr{O}_{\mathbb{P}^n}(-1)^{n+1}$ stands for the direct sum of $n+1$ copies of the twisting sheaf $\mathscr{O}_{\mathbb{P}^n}(-1)$.

Proof. Let us first construct the two morphisms $f\colon \Omega_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(-1)^{n+1}$ and $g\colon \mathscr{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathscr{O}_{\mathbb{P}^n}$ in the sequence. To motivate the definition of f, consider for $i,j \in \{0,\dots,n\}$ with $i \neq j$ the regular functions $\frac{x_i}{x_j}$ on $U_j := \{x \in \mathbb{P}^n : x_j \neq 0\} \subset \mathbb{P}^n$. If x_i and x_j were regular functions themselves, we would have by Remark 15.2

$$d\left(\frac{x_i}{x_j}\right) = -\frac{x_i}{x_j^2}\,dx_j + \frac{1}{x_j}\,dx_i. \qquad (*)$$

But of course x_i and x_j are not well-defined functions, so it seems that this equation does not make sense since dx_i and dx_j do not exist. However, if we denote the $n+1$ components of $\mathscr{O}_{\mathbb{P}^n}(-1)^{n+1}$ by the formal symbols $dx_0,\dots,dx_n$ we can use the idea of $(*)$ to *define* the morphism f by

$$f\colon \Omega_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(-1)^{n+1},\ d\left(\frac{x_i}{x_j}\right) \mapsto \Bigg(0,\dots,0,\underbrace{-\frac{x_i}{x_j^2}}_{\text{component } j},0,\dots,0,\underbrace{\frac{1}{x_j}}_{\text{component } i},0,\dots,0\Bigg).$$

In fact, as $d\left(\frac{x_i}{x_j}\right)$ for $i=0,\dots,n$ with $i \neq j$ generate $\Omega_{\mathbb{P}^n}|_{U_j}$ by Example 15.3 (a) this completely determines a morphism of sheaves of modules, and the standard rules of differentiation ensure that it is well-defined, i. e. that $d\left(\frac{x_i}{x_k}\right) = d\left(\frac{x_i}{x_j}\cdot\frac{x_j}{x_k}\right)$ and $\frac{x_i}{x_j}\,d\left(\frac{x_j}{x_k}\right) + \frac{x_j}{x_k}\,d\left(\frac{x_i}{x_j}\right)$ map to the same element — which is easily verified directly. Finally, the morphism g is simply defined by

$$g\colon \mathscr{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathscr{O}_{\mathbb{P}^n},\ (\varphi_0,\dots,\varphi_n) \mapsto \varphi_0 x_0 + \cdots + \varphi_n x_n.$$

It is now just straightforward commutative algebra to check that the sequence of the proposition is exact: By Lemma 13.21 we can do this on each U_j, so without loss of generality for $j=0$ on U_0, where $x_1,\dots,x_n$ are affine coordinates and $x_0 = 1$. By the above definition of the morphisms we have $f(dx_i) = -x_i\,dx_0 + dx_i$ in these coordinates, and hence the matrices over $K[x_1,\dots,x_n]$ corresponding by Lemma 14.7 to f and g are

$$A = \begin{pmatrix} -x_1 & \cdots & -x_n \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \quad\text{and}\quad B = \begin{pmatrix} 1 & x_1 & \cdots & x_n \end{pmatrix},$$

respectively. But for these matrices it is checked immediately that $\operatorname{Ker} A = \{0\}$, $\operatorname{Im} A = \operatorname{Ker} B$, and
26 $\operatorname{Im} B = K[x_1,\dots,x_n]$. □

Remark 15.9. Dualizing the Euler sequence of Proposition 15.8 (and noting by Exercise 13.25 that $\mathscr{O}_{\mathbb{P}^n}(-1)^\vee \cong \mathscr{O}_{\mathbb{P}^n}(1)$), we have an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0$$

that determines the tangent bundle of $\mathbb{P}^n$. In particular, for $\mathbb{P}^1$ we obtain the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \to T_{\mathbb{P}^1} \to 0.$$

Comparing this to Exercise 13.24 (where the first morphism is actually the same), we thus see that $T_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}(2)$.

In particular, every global section of the tangent bundle has exactly two zeros (counted with multiplicity), as we have already mentioned in Example 13.1. Over the complex numbers one can show that this is in fact a topological property: There is not even a *continuous* nowhere-zero tangent vector field on the real 2-dimensional unit sphere. This is usually called the "hairy ball theorem" and stated by saying that "one cannot comb a hedgehog (i. e. a ball) without a bald spot".

Proposition 15.10 (**Conormal sequence**). *Let X be a hypersurface of degree d in $\mathbb{P}^n$ over a field of characteristic* 0*. Then the cotangent sheaf of X is given by the exact sequence*

$$0 \to \mathscr{O}_X(-d) \to i^*\Omega_{\mathbb{P}^n} \to \Omega_X \to 0$$

on X, where $\mathscr{O}_X(-d)$ is as in Notation 14.19 and $i\colon X \to \mathbb{P}^n$ denotes the inclusion.

Proof. Let $I(X) = \langle f \rangle$ for a homogeneous polynomial f of degree d. The two maps in the sequence are

$$\mathscr{O}_X(-d) \to i^*\Omega_{\mathbb{P}^n},\ \varphi \mapsto d(f\varphi) \quad \text{and} \quad i^*\Omega_{\mathbb{P}^n} \to \Omega_X,\ d\varphi \mapsto d(\varphi|_X).$$

Note that the first map is well-defined as $f\varphi$ is a regular function if φ is a section of $\mathscr{O}_X(-d)$. We will show on an affine open cover that it is actually a morphism of sheaves of modules, and that the sequence is exact. Without loss of generality, it suffices to check this on $U_0 = \{x \in \mathbb{P}^n : x_0 \neq 0\}$, where we set $x_0 = 1$ and use $x_1, \dots, x_n$ as affine coordinates.

Note that $X \cap U_0$ is just the zero locus of the dehomogenization f^{i} on this open set. We set $R = K[x_1, \dots, x_n]$ and $S = R/\langle f^{\mathrm{i}} \rangle$. By the description of the pull-back and the cotangent sheaf in Construction 14.13 and Example 15.3, respectively, the sequence of S-modules corresponding to the given sequence of sheaves on U_0 is

$$0 \to S \to (R\,dx_1 \oplus \cdots \oplus R\,dx_n) \otimes_R S \to (S\,dx_1 \oplus \cdots \oplus S\,dx_n)/\langle df^{\mathrm{i}} \rangle \to 0,$$

or in other words

$$0 \to S \to S\,dx_1 \oplus \cdots \oplus S\,dx_n \to (S\,dx_1 \oplus \cdots \oplus S\,dx_n)/\langle df^{\mathrm{i}} \rangle \to 0, \qquad (*)$$

where the second non-trivial map is just the quotient, and the first is given by

$$\varphi \mapsto d(f^{\mathrm{i}}\varphi) = \underbrace{f^{\mathrm{i}}}_{=0 \text{ in } S} d\varphi + \varphi\, df^{\mathrm{i}} = \varphi\, df^{\mathrm{i}}.$$

Hence, this first map is the S-module homomorphism that is just multiplication with df^{i}. We therefore just have to prove its injectivity to see that the sequence $(*)$ is exact. So assume that we have an element $\varphi \in S$ with

$$\varphi\, df^{\mathrm{i}} = \varphi \frac{\partial f^{\mathrm{i}}}{\partial x_1} dx_1 + \cdots + \varphi \frac{\partial f^{\mathrm{i}}}{\partial x_n} dx_n \overset{!}{=} 0 \quad \in S\,dx_1 \oplus \cdots \oplus S\,dx_n,$$

i. e. that $\varphi \frac{\partial f^{\mathrm{i}}}{\partial x_k} \in \langle f^{\mathrm{i}} \rangle$ for all $k = 1, \dots, n$. As $\operatorname{char} K = 0$, at least one of these partial derivatives $\frac{\partial f^{\mathrm{i}}}{\partial x_k}$ must be non-zero. Moreover, f^{i} generates a radical ideal and hence has no repeated factors, and thus by the rules of differentiation $\frac{\partial f^{\mathrm{i}}}{\partial x_k}$ and f^{i} are coprime. Hence, $\varphi \frac{\partial f^{\mathrm{i}}}{\partial x_k} \in \langle f^{\mathrm{i}} \rangle$ requires $\varphi \in \langle f^{\mathrm{i}} \rangle$, i. e. $\varphi = 0 \in S$. This proves the injectivity of the first map in $(*)$, and thus that this sequence is exact. □

Remark 15.11. If X is a smooth hypersurface in $\mathbb{P}^n$ we can dualize the conormal sequence to compute the tangent bundle of X: We have the exact *normal sequence*

$$0 \to T_X \to i^* T_{\mathbb{P}^n} \to \mathscr{O}_X(d) \to 0$$

(where i^* commutes with dualizing $\Omega_{\mathbb{P}^n}$ since $\Omega_{\mathbb{P}^n}$ is locally free). Note that this sequence means that the fibers of the line bundle $\mathscr{O}_X(d)$ at a point $P \in X$ can be identified with the quotient $T_P\mathbb{P}^n / T_P X$, i. e. with the space of normal directions in $\mathbb{P}^n$ relative to X. This explains the name "normal sequence" resp. "conormal sequence" for the statement of Proposition 15.10; the line bundle $\mathscr{O}_X(d)$ is also called the *normal bundle* of X in $\mathbb{P}^n$.