10. Smooth Varieties

Let a be a point on a variety X. In the last chapter we have introduced the tangent cone C_aX as a way to study X locally around a (see Construction 9.20). By Corollary 9.24, it is a cone whose dimension is the local dimension $\operatorname{codim}_X\{a\}$ of X at a (or just $\dim X$ if X is irreducible), and we can think of it as the cone that best approximates X around a. In an affine open chart where a is the origin, we can compute C_aX by choosing an ideal with zero locus X and replacing each polynomial in this ideal by its initial term (Exercise 9.22 (b)).

However, in practice one often wants to approximate a given variety by a linear space rather than by a cone. We will therefore study now to what extent this is possible, and how the result compares to the tangent cones that we already know. Of course, the idea to construct this is just to take the *linear terms* instead of the *initial terms* of the defining polynomials when considering the origin in an affine variety. For simplicity, let us therefore assume for a moment that we have chosen an affine neighborhood of the point a such that $a = 0$ — we will see in Corollary 10.5 that the following construction actually does not depend on this choice.

Definition 10.1 (Tangent spaces)**.** Let a be a point on a variety X. By choosing an affine neighborhood of a we assume that $X \subset \mathbb{A}^n$ and that $a = 0$ is the origin, so that no polynomial $f \in I(X)$ has a constant term. Then

$$T_aX := V(f_1 : f \in I(X)) \quad \subset \mathbb{A}^n$$

is called the **tangent space** of X at a, where $f_1 \in K[x_1,\dots,x_n]$ denotes the linear term of a polynomial $f \in K[x_1,\dots,x_n]$ as in Definition 6.6 (a).

As in the case of tangent cones, we can consider T_aX either as an abstract variety (leaving its dimension as the only invariant since it is a linear space) or as a subspace of $\mathbb{A}^n$.

Remark 10.2.

(a) In contrast to the case of tangent cones in Exercise 9.22 (c), it always suffices in Definition 10.1 to take the zero locus only of the linear parts of a set S of generators for $I(X)$: If $f,g \in S$ are polynomials such that f_1 and g_1 vanish at a point $x \in \mathbb{A}^n$ then

$$(f+g)_1(x) = f_1(x) + g_1(x) = 0$$

$$\text{and} \quad (hf)_1(x) = h(0)\,f_1(x) + f(0)\,h_1(x) = h(0)\cdot 0 + 0\cdot h_1(x) = 0$$

for an arbitrary polynomial $h \in K[x_1,\dots,x_n]$, and hence $x \in T_aX$.

(b) However, again in contrast to the case of tangent cones in Exercise 9.22 it is crucial in Definition 10.1 that we take the radical ideal of X and not just any ideal with zero locus X: The ideals $\langle x\rangle$ and $\langle x^2\rangle$ in $K[x]$ have the same zero locus $\{0\}$ in $\mathbb{A}^1$, but the zero locus of the linear term of x is the origin again, whereas the zero locus of the linear term of x^2 is all of $\mathbb{A}^1$.

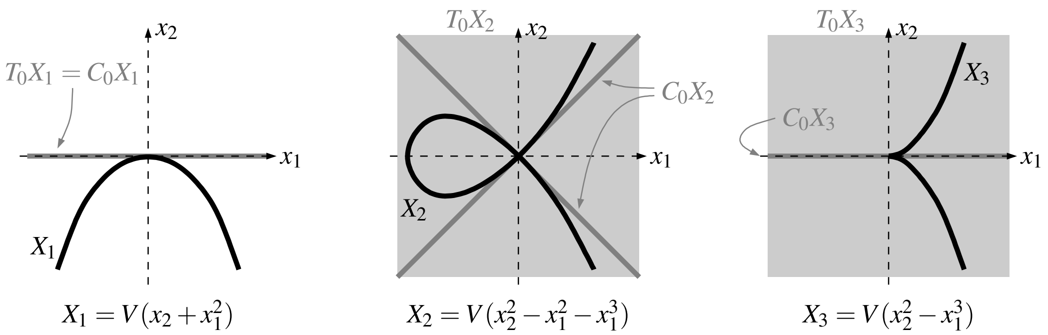
(c) For polynomials vanishing at the origin, a non-vanishing linear term is clearly always initial. Hence by Exercise 9.22 (b) it follows that $C_aX \subset T_aX$, i.e. that the tangent space always contains the tangent cone. In particular, this means by Corollary 9.24 that $\dim T_aX \geq \operatorname{codim}_X\{a\}$.

Example 10.3. Consider again the three complex curves X_1, X_2, X_3 of Example 9.21. By taking the initial resp. linear term of the defining polynomials we can compute the tangent cones and spaces of these curves at the origin:

- $X_1 = V(x_2 + x_1^2)$: $C_0X_1 = T_0X_1 = V(x_2)$;
- $X_2 = V(x_2^2 - x_1^2 - x_1^3)$: $C_0X_2 = V(x_2^2 - x_1^2)$, $T_0X_2 = V(0) = \mathbb{A}^2$;

- $X_3 = V(x_2^2 - x_1^3)$: $C_0X_3 = V(x_2^2) = V(x_2)$, $T_0X_3 = V(0) = \mathbb{A}^2$.

The following picture shows these curves together with their tangent cones and spaces. Note that for the curve X_1 the tangent cone is already a linear space, and the notions of tangent space and tangent cone agree. In contrast, the tangent cone of X_2 at the origin is not linear. By Remark 10.2 (c), the tangent space T_0X_2 must be a linear space containing C_0X_2, and hence it is necessarily all of $\mathbb{A}^2$. However, the curve X_3 shows that the tangent space is not always the linear space spanned by the tangent cone.



Before we study the relation between tangent spaces and cones in more detail, let us show first of all that the (dimension of the) tangent space is actually an intrinsic local invariant of a variety around a point, i. e. that it does not depend on a choice of affine open subset or coordinates around the point. We will do this by establishing an alternative description of the tangent space that does not need any such choices. The key observation needed for this is the isomorphism of the following lemma, in which for clarity we denote by $\overline{f} \in A(X) = K[x_1,\dots,x_n]/I(X)$ the class of a polynomial $f \in K[x_1,\dots,x_n]$ modulo the ideal of an affine variety X.

Lemma 10.4. *Let $X \subset \mathbb{A}^n$ be an affine variety containing the origin $a = 0$, whose ideal is then $I(a) = \langle \overline{x_1},\dots,\overline{x_n} \rangle \trianglelefteq A(X)$. Then there is a natural vector space isomorphism*

$$I(a)/I(a)^2 \cong \operatorname{Hom}_K(T_aX, K).$$

In other words, the tangent space T_aX is naturally the vector space dual to $I(a)/I(a)^2$.

Proof. Consider the K-linear map

$$\varphi\colon I(a) \to \operatorname{Hom}_K(T_aX, K),\ \overline{f} \mapsto f_1|_{T_aX}$$

sending the class of a polynomial to its linear term, regarded as a map restricted to the tangent space. By definition of the tangent space, this map is well-defined. Moreover, note that φ is surjective since any linear map on T_aX can be extended to a linear map on $\mathbb{A}^n$. So by the homomorphism theorem it suffices to prove that $\operatorname{Ker}\varphi = I(a)^2$.

"$\subset$" Consider the vector subspace $W = \{g_1 : g \in I(X)\}$ of $K[x_1,\dots,x_n]$, and let k be its dimension. Then its zero locus T_aX has dimension $n-k$, and hence the space of linear forms vanishing on T_aX has dimension k again. As it clearly contains W, we conclude that W must be equal to the space of linear forms vanishing on T_aX.

So if $\overline{f} \in \operatorname{Ker}\varphi$, i. e. the linear term of f vanishes on T_aX, we know that there is a polynomial $g \in I(X)$ with $g_1 = f_1$. But then $f-g$ has no constant or linear term, and hence we have $\overline{f} = \overline{f-g} \in I(a)^2$.

"$\supset$" If $\overline{f},\overline{g} \in I(a)$ then $(fg)_1 = f(0)\,g_1 + g(0)\,f_1 = 0 \cdot g_1 + 0 \cdot f_1 = 0$, and hence $\varphi(\overline{fg}) = 0$. □

In order to make Lemma 10.4 into an intrinsic description of the tangent space we need to transfer it from the affine coordinate ring $A(X)$ (which for a general variety would require the choice of an affine coordinate chart that sends the given point a to the origin) to the local ring $\mathscr{O}_{X,a}$ (which is independent of any choices). To do this, recall by Lemma 3.19 that with the notations from above we have $\mathscr{O}_{X,a} \cong S^{-1}A(X)$, where $S = A(X)\backslash I(a)$ is the multiplicatively closed subset of polynomial functions that are non-zero at the point a. In this ring

$$S^{-1}I(a) = \left\{ \frac{g}{f} : g, f \in A(X) \text{ with } g(a) = 0 \text{ and } f(a) \neq 0 \right\}$$

is just the maximal ideal I_a of all local functions vanishing at a. Using these constructions we obtain the following result.

Corollary 10.5. *With notations as in Lemma 10.4 and $S = A(X)\backslash I(a)$ we have*

$$I(a)/I(a)^2 \;\cong\; (S^{-1}I(a))/(S^{-1}I(a))^2.$$

In particular, if $a \in X$ is a point on any variety X and I_a is the unique maximal ideal in $\mathscr{O}_{X,a}$, then T_aX is naturally isomorphic to the vector space dual to I_a/I_a^2, and thus independent of any choices.

Proof. For an element $f \in S$ we have $f(a) \neq 0$, so (the class of) f is non-zero and hence invertible in $A(X)/I(a) \cong K$. In other words, we have $\frac{1}{f} = c$ in $A(X)/I(a)$ for some $c \in K$ (in fact, for $c = \frac{1}{f(a)}$). But then we have $\frac{g}{f} = cg \in I(a)/I(a)^2$ for all $g \in I(a)/I(a)^2$, which means that localizing at S does not change $I(a)/I(a)^2$ since the elements of S are already invertible. We thus see that

$$I(a)/I(a)^2 \cong S^{-1}\big(I(a)/I(a)^2\big) \cong S^{-1}I(a)/(S^{-1}I(a))^2,$$

where the last isomorphism holds since localization commutes with quotients [G3, Corollary 6.22 (b)]. □

In fact, because of this result the tangent space T_aX is often *defined* to be the vector space dual to I_a/I_a^2.

Exercise 10.6. Let $f\colon X \to Y$ be a morphism of varieties, and let $a \in X$. Show that f induces a linear map $T_aX \to T_{f(a)}Y$ between tangent spaces.

We have now constructed two objects associated to the local structure of a variety X at a point $a \in X$:

- the *tangent cone* C_aX, which is a cone of dimension $\operatorname{codim}_X\{a\}$, but in general not a linear space; and
- the *tangent space* T_aX, which is a linear space, but whose dimension might be bigger than $\operatorname{codim}_X\{a\}$.

Of course, we should give special attention to the case when these two notions agree, i. e. when X can be approximated around a by a linear space whose dimension is the local dimension of X at a. This defines the so-called notion of *smoothness*; in fact we will see in Example 10.12 that it agrees with the concept of smoothness for plane curves that we have already introduced in [G2, Definition 2.20].

Definition 10.7 (Smooth and singular varieties)**.** Let X be a variety.

(a) A point $a \in X$ is called **smooth**, **regular**, or **non-singular** if $T_aX = C_aX$. Otherwise it is called a **singular** point of X.

(b) If X has a singular point we say that X is singular. Otherwise X is called smooth, regular, or non-singular.

Example 10.8. Of the three curves of Example 10.3, exactly the first one is smooth at the origin. As in our original motivation for the definition of tangent spaces, this is just the statement that X_1 can be approximated around the origin by a straight line — in contrast to X_2 and X_3, which have a "multiple point" resp. a "corner" there. A more precise geometric interpretation of smoothness can be obtained by comparing our algebraic situation with the Implicit Function Theorem from analysis, see Remark 10.15.

Lemma 10.9. *Let X be a variety, and let $a \in X$ be a point. The following statements are equivalent:*

(a) *The point a is smooth on X.*

(b) $\dim T_a X = \operatorname{codim}_X\{a\}$.

(c) $\dim T_a X \leq \operatorname{codim}_X\{a\}$.

Proof. The implication (a) ⇒ (b) follows immediately from Corollary 9.24. To prove (b) ⇒ (a), recall by Remark 10.2 (c) that the tangent space $T_a X$ contains the tangent cone $C_a X$, which is also of dimension $\operatorname{codim}_X\{a\}$ by Corollary 9.24. As $T_a X$ is irreducible (since it is a linear space), this is only possible if $T_a X = C_a X$, i. e. if a is a smooth point of X.

Finally, the equivalence of (b) and (c) follows since the inequality $\dim T_a X \geq \operatorname{codim}_X\{a\}$ always holds by Remark 10.2 (c). □

Remark 10.10 (Smoothness in commutative algebra)**.** Let a be a point on a variety X.

(a) Let $I_a \trianglelefteq \mathscr{O}_{X,a}$ be the maximal ideal as in Lemma 3.19. Combining Corollary 10.5 with Lemma 10.9 we see that a is a smooth point of X if and only if the vector space dimension of I_a/I_a^2 is equal to the local dimension $\operatorname{codim}_X\{a\}$ of X at a. This is a property of the local ring $\mathscr{O}_{X,a}$ alone, and one can therefore study it with methods from commutative algebra. A ring with this property is usually called a *regular local ring* [G3, Definition 11.38], which is also the reason for the name “regular point” in Definition 10.7 (a).

(b) It is a result of commutative algebra that a regular local ring as in (a) is always an integral domain [G3, Proposition 11.40]. Translating this into geometry as in Proposition 2.8, this yields the intuitively obvious statement that a variety is *locally irreducible* at every smooth point a, i. e. that X has only one irreducible component meeting a. Equivalently, any point on a variety at which two irreducible components meet is necessarily a singular point.

The good thing about smoothness is that it is very easy to check using (formal) partial derivatives:

Proposition 10.11 (Affine Jacobi criterion). *Let $a \in X$ be a point on an affine variety $X \subset \mathbb{A}^n$, and let $I(X) = \langle f_1, \dots, f_r \rangle$. Then X is smooth at a if and only if the rank of the $r \times n$* ***Jacobian matrix***

$$\left(\frac{\partial f_i}{\partial x_j}(a) \right)_{i,j}$$

is at least $n - \operatorname{codim}_X\{a\}$, and in this case the rank of this matrix is in fact equal to $n - \operatorname{codim}_X\{a\}$.

Proof. Let $x = (x_1, \dots, x_n)$ be the coordinates of $\mathbb{A}^n$, and let $y := x - a$ be the shifted coordinates in which the point a becomes the origin. By a formal Taylor expansion, the linear term of the polynomial f_i in these coordinates y is $\sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(a) \cdot y_j$. Hence the tangent space $T_a X$ is by Definition 10.1 and Remark 10.2 (a) the zero locus of these linear terms, i. e. the kernel of the Jacobian matrix $J = \left(\frac{\partial f_i}{\partial x_j}(a) \right)_{i,j}$. So by Lemma 10.9 “(a) ⇔ (c)” the point a is smooth on X if and only we have if $\dim \operatorname{Ker} J \leq \operatorname{codim}_X\{a\}$, which is equivalent to $\operatorname{rk} J \geq n - \operatorname{codim}_X\{a\}$. The part “(a) ⇔ (b)” of Lemma 10.9 then shows that we have in fact equality in this case. □

Example 10.12. Let $X \subset \mathbb{A}^2$ be a plane curve with ideal $I(X) = \langle f \rangle$. By Proposition 10.11, the point a is then smooth on X if and only if the rank of the Jacobian matrix $\left(\frac{\partial f}{\partial x_1} \; \frac{\partial f}{\partial x_2} \right)$ at a is (at least) 1, i. e. if and only if at least one of these two derivatives is non-zero at a. This is precisely the criterion from [G2, Proposition 2.24], showing that our new definition agrees with the old one for plane curves.

To check smoothness for a point on a projective variety, we can of course restrict to an affine open subset of the point. However, the following exercise shows that there is also a projective version of the Jacobi criterion that does not need these affine patches and works directly with the homogeneous coordinates instead.

Exercise 10.13.

(a) Show that

$$\sum_{i=0}^n x_i \cdot \frac{\partial f}{\partial x_i} = d \cdot f$$

for any homogeneous polynomial $f \in K[x_0,\dots,x_n]$ of degree d.

(b) (**Projective Jacobi criterion**) Let $X \subset \mathbb{P}^n$ be a projective variety with homogeneous ideal $I(X) = \langle f_1,\dots,f_r \rangle$, and let $a \in X$. Prove that X is smooth at a if and only if the rank of the $r \times (n+1)$ Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j}$ is at least $n - \operatorname{codim}_X\{a\}$.

In this criterion, note that the entries $\frac{\partial f_i}{\partial x_j}(a)$ of the Jacobian matrix are not well-defined: Multiplying the coordinates of a by a scalar $\lambda \in K^*$ will multiply $\frac{\partial f_i}{\partial x_j}(a)$ by $\lambda^{d_i - 1}$, where d_i is the degree of f_i. However, these are just row transformations of the Jacobian matrix, which do not affect its rank. Hence the condition in the projective Jacobi criterion is well-defined.

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Recall that the Jacobi criterion of Proposition 10.11 requires generators for the ideal $I(X)$ of an affine variety X in order to determine if X is smooth. However, in case one only knows polynomials $f_1,\dots,f_r$ with $V(f_1,\dots,f_r) = X$ (whose ideal might not be radical) one direction of the Jacobi criterion still holds and gives rise to the following important variants of the proposition. They also hold in the projective setting of Exercise 10.13 (b).

Corollary 10.14 (Variants of the Jacobi criterion)**.** *Let $f_1,\dots,f_r \in K[x_1,\dots,x_n]$ be polynomials, and let $a \in X := V(f_1,\dots,f_r) \subset \mathbb{A}^n$.*

(a) *If* $\operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j} \geq n - \operatorname{codim}_X\{a\}$ *then X is smooth at a.*

(b) *If* $\operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j} = r$ *(i. e. the Jacobian matrix has maximal row rank) then X is smooth at a of local dimension $n - r$.*

Proof.

(a) As $f_1,\dots,f_r \in I(X)$, we can extend these polynomials to a set of generators $f_1,\dots,f_s$ of $I(X)$ for some $s \geq r$. Then

$$\operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{\substack{1\le i\le s\\ 1\le j\le n}} \geq \operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{\substack{1\le i\le r\\ 1\le j\le n}} \geq n - \operatorname{codim}_X\{a\},$$

and so the claim follows from Proposition 10.11.

(b) As every irreducible component of X has dimension at least $n - r$ by Proposition 2.28 (c) we know that $\operatorname{codim}_X\{a\} \geq n - r$. Hence by assumption the rank of the Jacobian matrix is $r \geq n - \operatorname{codim}_X\{a\}$, and so X is smooth at a by (a). Moreover, its local dimension is then $\operatorname{codim}_X\{a\} = n - r$ by the equality part of Proposition 10.11. □

Remark 10.15 (Relation to the Implicit Function Theorem)**.** The version of the Jacobi criterion of Corollary 10.14 (b) is closely related to the *Implicit Function Theorem* from analysis. Given real polynomials $f_1,\dots,f_r \in \mathbb{R}[x_1,\dots,x_n]$ (or more generally continuously differentiable functions on an open subset of $\mathbb{R}^n$) and a point a in their common zero locus $X = V(f_1,\dots,f_r)$ such that the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j}$ has rank r, this theorem states roughly that X is locally around a the graph of a continuously differentiable function [G1, Proposition 27.9] — so that in particular it does not have any "corners". It can be shown that the same result holds over the complex numbers as well. So in the case $K = \mathbb{C}$ the statement of Corollary 10.14 (b) that X is smooth at a can also be interpreted in this geometric way.

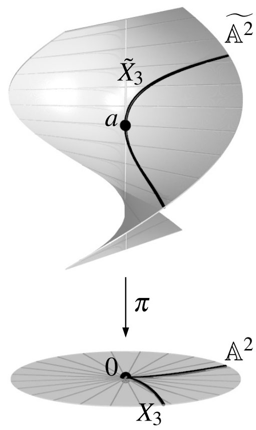
Note however that there is no algebraic analogue of the Implicit Function Theorem itself: For example, the polynomial equation $f(x_1,x_2) := x_2 - x_1^2 = 0$ cannot be solved for x_1 by a *regular* function locally around the point $(1,1)$, although $\frac{\partial f}{\partial x_1}(1,1) = -2 \neq 0$ — it can only be solved by a continuously differentiable function $x_1 = \sqrt{x_2}$.

Example 10.16. Consider again the curve $X_3 = V(x_2^2 - x_1^3) \subset \mathbb{A}^2_{\mathbb{C}}$ of Examples 9.21 and 10.3. The Jacobian matrix of the single polynomial $f = x_2^2 - x_1^3$ is

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix} = (-3x_1^2 \;\; 2x_2),$$

so it has rank (at least) $2 - \dim X = 1$ exactly if $(x_1, x_2) \in \mathbb{A}^2 \backslash \{0\}$. Hence the Jacobi criterion does not only reprove our observation from Example 10.3 that the origin is a singular point of X_3, but also shows simultaneously that all other points of X_3 are smooth.

In the picture on the right we have also drawn the blow-up $\tilde{X}_3$ of X_3 at its singular point again. We have seen already that its exceptional set consists of only one point $a \in \tilde{X}_3$. Let us now check that this is a smooth point of $\tilde{X}_3$ — as we would expect from the picture.



In the coordinates $((x_1,x_2),(y_1:y_2))$ of $\tilde{X}_3 \subset \widetilde{\mathbb{A}^2} \subset \mathbb{A}^2 \times \mathbb{P}^1$, the point a is given as $((0,0),(1:0))$. So around a we can use the affine open chart $U_1 = \{((x_1,x_2),(y_1:y_2)) : y_1 \neq 0\}$ with affine coordinates x_1 and y_2 as in Example 9.15. By Exercise 9.22 (a), the blow-up $\tilde{X}_3$ is given in these coordinates by

$$\frac{(x_1 y_2)^2 - x_1^3}{x_1^2} = 0, \quad \text{i. e. } g(x_1,y_2) := y_2^2 - x_1 = 0.$$

As the Jacobian matrix

$$\begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial y_2} \end{pmatrix} = (-1 \;\; 2y_2)$$

of this polynomial has rank 1 at every point, the Jacobi criterion tells us that $\tilde{X}_3$ is smooth. In fact, from the defining equation $y_2^2 - x_1 = 0$ we see that on the open subset U_1 the curve $\tilde{X}_3$ is just the "standard parabola" tangent to the exceptional set of $\widetilde{\mathbb{A}^2}$ (which is given on U_1 by the equation $x_1 = 0$ by the proof of Proposition 9.23).

It is actually a general statement that blowing up makes singular points "nicer", and that successive blow-ups will eventually make all singular points smooth. This process is called *resolution of singularities*. We will not discuss this here in detail, but the following exercise shows an example of this process.

Exercise 10.17. For $k \in \mathbb{N}$ let X_k be the complex affine curve $X_k := V(x_2^2 - x_1^{2k+1}) \subset \mathbb{A}^2_{\mathbb{C}}$. Show that X_k is not isomorphic to X_l if $k \neq l$.

(Hint: Consider the blow-up of X_k at the origin.)

Exercise 10.18. Let $X \subset \mathbb{P}^3$ be the degree-3 Veronese embedding of $\mathbb{P}^1$ as in Exercise 7.31. Of course, X must be smooth since it is isomorphic to $\mathbb{P}^1$. Verify this directly using the projective Jacobi criterion of Exercise 10.13 (b).

Corollary 10.19. *The set of smooth points of a variety is open.*

Proof. Let a be a smooth point on a variety X; we have to find a smooth open neighborhood of this point.

By restricting to an affine open set, we may first of all assume that $X \subset \mathbb{A}^n$ is affine. Next, by possibly restricting to a smaller affine subset, we may assume by Remark 10.10 (b) that X is irreducible, so that its local dimension is constant at all points, equal to $\dim X$. Then by the Jacobi criterion of Proposition 10.11 we know that the smooth locus of X is exactly the set of points at which the rank of the Jacobian matrix of generators of $I(X)$ is at least $n - \dim X$. As this is an open condition (given by the non-vanishing of at least one minor of size $n - \dim X$), the result follows. □

Remark 10.20 (Generic smoothness)**.** Corollary 10.19 tells us by a simple application of the Jacobi criterion that the locus of smooth points on a variety is open, but it does not guarantee that it is non-empty. It is much harder to show that the smooth locus is in fact always dense — a statement referred to as "*generic smoothness*" [H, Theorem I.5.3]. We will prove this here only in the case of a hypersurface $X = V(f) \subset \mathbb{A}^n$ for a non-constant irreducible polynomial $f \in K[x_1,\dots,x_n]$. As X is then irreducible, it suffices to see that the set of smooth points of X is non-empty (since a non-empty open subset in an irreducible space is always dense by Remark 2.16).

Assume the contrary, i. e. that all points of X are singular. Then by Proposition 10.11 the Jacobian matrix of f must have rank 0 at every point, which means that $\frac{\partial f}{\partial x_i}(a) = 0$ for all $a \in X$ and $i = 1,\dots,n$. Hence $\frac{\partial f}{\partial x_i} \in I(V(f)) = \langle f \rangle$ by the Nullstellensatz. But since f is irreducible and the polynomial $\frac{\partial f}{\partial x_i}$ has smaller degree than f this is only possible if $\frac{\partial f}{\partial x_i} = 0$ for all i.

In the case $\operatorname{char} K = 0$ this is already a contradiction to f being non-constant. If $\operatorname{char} K = p$ is positive, then f must be a polynomial in $x_1^p,\dots,x_n^p$, and so

$$f = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} x_1^{pi_1} \cdot \dots \cdot x_n^{pi_n} = \left(\sum_{i_1,\dots,i_n} b_{i_1,\dots,i_n} x_1^{i_1} \cdot \dots \cdot x_n^{i_n} \right)^p,$$

for p-th roots $b_{i_1,\dots,i_n}$ of $a_{i_1,\dots,i_n}$. This is a contradiction since f was assumed to be irreducible.

Example 10.21 (Fermat hypersurfaces)**.** For given $n,d \in \mathbb{N}_{>0}$ consider the *Fermat hypersurface*

$$X := V_p(x_0^d + \dots + x_n^d) \quad \subset \mathbb{P}^n.$$

We want to show that X is smooth for all choices of n, d, and K. For this we use the Jacobian matrix $(dx_0^{d-1} \ \cdots \ dx_n^{d-1})$ of the given polynomial:

(a) If $\operatorname{char} K \nmid d$ the Jacobian matrix has rank 1 at every point, so X is smooth by Exercise 10.13 (b).

(b) If $p = \operatorname{char} K \mid d$ we can write $d = k\,p^r$ for some $r \in \mathbb{N}_{>0}$ and $p \nmid k$. Since

$$x_0^d + \dots + x_n^d = (x_0^k + \dots + x_n^k)^{p^r},$$

we see again that $X = V_p(x_0^k + \dots + x_n^k)$ is smooth by (a).

Exercise 10.22. Let X be a projective variety of dimension n. Prove:

(a) There is an injective morphism $X \to \mathbb{P}^{2n+1}$.

(b) There is in general no such morphism that is an isomorphism onto its image.

Exercise 10.23. Let $n \geq 2$. Prove:

(a) Every smooth hypersurface in $\mathbb{P}^n$ is irreducible.

(b) A general hypersurface in $\mathbb{P}^n_{\mathbb{C}}$ is smooth (and thus by (a) irreducible). More precisely, for a given $d \in \mathbb{N}_{>0}$ the vector space $\mathbb{C}[x_0,\dots,x_n]_d$ has dimension $\binom{n+d}{n}$, and so the space of all homogeneous degree-d polynomials in $x_0,\dots,x_n$ modulo scalars can be identified with the projective space $\mathbb{P}_{\mathbb{C}}^{\binom{n+d}{n}-1}$. Show that the subset of this projective space of all (classes of) polynomials f such that f is irreducible and $V_p(f)$ is smooth is dense and open.

Exercise 10.24 (Dual curves)**.** Assume that $\operatorname{char} K \neq 2$, and let $f \in K[x_0,x_1,x_2]$ be a homogeneous polynomial whose partial derivatives $\frac{\partial f}{\partial x_i}$ for $i = 0,1,2$ do not vanish simultaneously at any point of $X = V_p(f) \subset \mathbb{P}^2$. Then the image of the morphism

$$F \colon X \to \mathbb{P}^2,\ a \mapsto \left(\frac{\partial f}{\partial x_0}(a) : \frac{\partial f}{\partial x_1}(a) : \frac{\partial f}{\partial x_2}(a) \right)$$

is called the *dual curve* to X.

(a) Find a geometric description of F. What does it mean geometrically if $F(a) = F(b)$ for two distinct points $a,b \in X$?

(b) If X is a conic, prove that its dual $F(X)$ is also a conic.

(c) For any five lines in $\mathbb{P}^2$ in general position (what does this mean?) show that there is a unique conic in $\mathbb{P}^2$ that is tangent to all of them.