1. Affine Varieties

As explained in the introduction, the goal of algebraic geometry is to study solutions of polynomial equations in several variables over a fixed ground field, so we will start by making the corresponding definitions. In order to make the correspondence between geometry and algebra as clear as possible (in particular to allow for Hilbert's Nullstellensatz in Proposition 1.10) we will restrict to the case of algebraically closed ground fields first. In particular, if we draw pictures over $\mathbb{R}$ they should always be interpreted as the real points of an underlying complex situation.

Convention 1.1.

(a) In the following, until we introduce schemes in Chapter 12, K will always denote a fixed algebraically closed ground field (such as e. g. $K = \mathbb{C}$).

(b) Rings are always assumed to be commutative with a multiplicative unit 1. If J is an ideal in a ring R we will write this as $J \trianglelefteq R$, and denote the *radical* of J by

$$\sqrt{J} := \{f \in R : f^k \in J \text{ for some } k \in \mathbb{N}\}.$$

The ideal generated by a subset S of a ring will be written as $\langle S \rangle$.

(c) As usual, we will denote the polynomial ring in n variables $x_1, \dots, x_n$ over K by $K[x_1, \dots, x_n]$, and the value of a polynomial $f \in K[x_1, \dots, x_n]$ at a point $a = (a_1, \dots, a_n) \in K^n$ by $f(a)$. If there is no risk of confusion we will often denote a point in K^n by the same letter x as we used for the formal variables, writing $f \in K[x_1, \dots, x_n]$ for the polynomial and $f(x)$ for its value at a point $x \in K^n$. The *degree* of a non-zero polynomial f is always meant to be its total degree, i. e. the biggest integer $d \in \mathbb{N}$ such that f contains a non-zero monomial $x_1^{i_1} \cdots x_n^{i_n}$ with $i_1 + \cdots + i_n = d$.

Definition 1.2 (Affine varieties)**.**

(a) We call

$$\mathbb{A}^n := \mathbb{A}^n_K := \{(a_1, \dots, a_n) : a_i \in K \text{ for } i = 1, \dots, n\}$$

the **affine n-space** over K.

Note that $\mathbb{A}^n_K$ is just K^n as a set. It is customary to use two different notations here since K^n is also a K-vector space and a ring. We will usually use the notation $\mathbb{A}^n_K$ if we want to ignore these additional structures: For example, addition and scalar multiplication are defined on K^n, but not on $\mathbb{A}^n_K$. The affine space $\mathbb{A}^n_K$ will be the ambient space for our zero loci of polynomials below.

(b) For a subset $S \subset K[x_1, \dots, x_n]$ of polynomials we call

$$V(S) := \{x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in S\} \quad \subset \mathbb{A}^n$$

the (affine) **zero locus** of S. Subsets of $\mathbb{A}^n$ of this form are called **(affine) varieties**. If $S = \{f_1, \dots, f_k\}$ is a finite set, we will write $V(S) = V(\{f_1, \dots, f_k\})$ also as $V(f_1, \dots, f_k)$.

Remark 1.3. Some authors refer to zero loci of polynomials in $\mathbb{A}^n$ as in Definition 1.2 (b) as *(affine) algebraic sets*, reserving the name "affine variety" for such zero loci that are in addition *irreducible* (a concept that we will introduce in Definition 2.5 (a)).

Lemma 1.4.

(a) *For any $S_1 \subset S_2 \subset K[x_1, \dots, x_n]$ we have $V(S_1) \supset V(S_2)$.*

(b) *For any $S_1, S_2 \subset K[x_1, \dots, x_n]$ we have $V(S_1) \cup V(S_2) = V(S_1 S_2)$, where as usual we set $S_1 S_2 := \{fg : f \in S_1, g \in S_2\}$.*

(c) *If J is any index set and $S_i \subset K[x_1,\dots,x_n]$ for all $i \in J$ then $\bigcap_{i\in J} V(S_i) = V(\bigcup_{i\in J} S_i)$.*

In particular, finite unions and arbitrary intersections of affine varieties are again affine varieties.

Proof.

(a) If $x \in V(S_2)$, i.e. $f(x) = 0$ for all $f \in S_2$, then in particular $f(x) = 0$ for all $f \in S_1$, and hence $x \in V(S_1)$.

(b) "$\subset$" If $x \in V(S_1) \cup V(S_2)$ then $f(x) = 0$ for all $f \in S_1$ or $g(x) = 0$ for all $g \in S_2$. In any case this means that $(fg)(x) = 0$ for all $f \in S_1$ and $g \in S_2$, i.e. that $x \in V(S_1S_2)$.

"$\supset$" If $x \notin V(S_1) \cup V(S_2)$, i.e. $x \notin V(S_1)$ and $x \notin V(S_2)$, then there are $f \in S_1$ and $g \in S_2$ with $f(x) \neq 0$ and $g(x) \neq 0$. Then $(fg)(x) \neq 0$, and hence $x \notin V(S_1S_2)$.

(c) We have $x \in \bigcap_{i\in J} V(S_i)$ if and only if $f(x) = 0$ for all $f \in S_i$ for all $i \in J$, which is the case if and only if $x \in V(\bigcup_{i\in J} S_i)$. □

Example 1.5. Some simple examples of affine varieties are:

(a) The affine n-space itself is an affine variety, since $\mathbb{A}^n = V(0)$. Similarly, the empty set $\emptyset = V(1)$ is an affine variety.

(b) Any point in $a = (a_1,\dots,a_n) \in \mathbb{A}^n$ is an affine variety, since $\{a\} = V(x_1 - a_1,\dots,x_n - a_n)$. By Lemma 1.4 (b), finite subsets of $\mathbb{A}^n$ are then affine varieties as well.

In the special case of $\mathbb{A}^1$, note that the zero locus of any non-zero polynomial in $K[x_1]$ is already finite. Hence the affine varieties in $\mathbb{A}^1$ are exactly $\mathbb{A}^1$ itself and all finite sets.

(c) Linear subspaces of $\mathbb{A}^n = K^n$ are affine varieties.

(d) If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine varieties then so is the *product* $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$.

Remark 1.6 (Affine varieties are zero loci of ideals)**.** Let f and g be polynomials that vanish on a certain subset $X \subset \mathbb{A}^n$. Then $f + g$ and hf for any polynomial h clearly vanish on X as well. This means that the set $S \subset K[x_1,\dots,x_n]$ defining an affine variety $X = V(S)$ is certainly not unique: For any $f,g \in S$ and any polynomial h we can add $f + g$ and hf to S without changing its zero locus, so that we always have $V(\langle S\rangle) = V(S)$. In particular, any affine variety in $\mathbb{A}^n$ can be written as the zero locus of an ideal in $K[x_1,\dots,x_n]$.

As any ideal in $K[x_1,\dots,x_n]$ is finitely generated by *Hilbert's Basis Theorem* [G3, Proposition 7.13 and Remark 7.15], this means moreover that any affine variety can be written as the zero locus of finitely many polynomials.

By Remark 1.6 we often can (and will) restrict our attention to zero loci of ideals. Let us therefore reformulate the results of Lemma 1.4 in terms of standard ideal-theoretic operations.

Lemma 1.7 (Properties of $V(\cdot)$)**.** *For any ideals J, J_1, J_2 in $K[x_1,\dots,x_n]$ we have*

(a) $V(\sqrt{J}) = V(J)$;

(b) $V(J_1) \cup V(J_2) = V(J_1J_2) = V(J_1 \cap J_2)$;

(c) $V(J_1) \cap V(J_2) = V(J_1 + J_2)$.

Proof.

(a) The inclusion "$\subset$" follows directly from Lemma 1.4 (a) since $\sqrt{J} \supset J$. For the other inclusion assume that $x \in V(J)$ and $f \in \sqrt{J}$. Then $f^k \in J$ for some $k \in \mathbb{N}$, so that $f^k(x) = 0$, and hence also $f(x) = 0$. This means that $x \in V(\sqrt{J})$.

(b) The first equation is just a reformulation of Lemma 1.4 (b), the second then follows from (a) since $\sqrt{J_1J_2} = \sqrt{J_1 \cap J_2}$ [G3, Lemma 1.7 (b)].

(c) This follows from Lemma 1.4 (c) together with Remark 1.6 since $J_1 + J_2$ is just the ideal generated by $J_1 \cup J_2$. □

Remark 1.6 is important since it is in some sense the basis of algebraic geometry: It relates *geometric objects* (affine varieties) to *algebraic objects* (ideals). In fact, it will be the main goal of this first chapter to study this correspondence in detail. We have already assigned affine varieties to ideals in Definition 1.2 (b) and Remark 1.6, so let us now introduce an operation that does the opposite job.

Definition 1.8 (Ideal of a subset of $\mathbb{A}^n$)**.** Let $X \subset \mathbb{A}^n$ be any subset. Then

$$I(X) := \{f \in K[x_1,\dots,x_n] : f(x) = 0 \text{ for all } x \in X\}$$

is called the **ideal** of X (note that this is indeed an ideal by Remark 1.6).

Remark 1.9.

(a) In analogy to Lemma 1.4 (a), it is obvious that the ideal of a subset reverses inclusions as well: If $X_1 \subset X_2 \subset \mathbb{A}^n$ then $I(X_1) \supset I(X_2)$.

(b) Note that $I(X)$ is always a radical ideal: If $f^k \in I(X)$ for some $f \in K[x_1,\dots,x_n]$ and $k \in \mathbb{N}$, then $f^k(x) = 0$ for all $x \in X$, hence $f(x) = 0$ for all $x \in X$, and thus $f \in I(X)$.

Hence, the operation $I(\cdot)$ of Definition 1.8 maps an affine variety to a radical ideal, and the operation $V(\cdot)$ of Definition 1.2 (b) can map a radical ideal back to an affine variety. It is a central result of commutative algebra that this is actually a bijection. Let us briefly recall this result, which even in the English literature is usually referred to by its German name as *Hilbert's Nullstellensatz* ("theorem of the zeros").

Proposition 1.10 (Hilbert's Nullstellensatz).

(a) *For any affine variety $X \subset \mathbb{A}^n$ we have $V(I(X)) = X$.*

(b) *For any ideal $J \trianglelefteq K[x_1,\dots,x_n]$ we have $I(V(J)) = \sqrt{J}$.*

In particular, there is an inclusion-reversing bijection

$$\begin{array}{rcl} \{\text{affine varieties in } \mathbb{A}^n\} & \stackrel{1:1}{\longleftrightarrow} & \{\text{radical ideals in } K[x_1,\dots,x_n]\} \\ X & \longmapsto & I(X) \\ V(J) & \longleftarrow\!\shortmid & J. \end{array}$$

Proof. Three of the four inclusions of (a) and (b) are actually easy:

(a) "$\supset$": If $x \in X$ then $f(x) = 0$ for all $f \in I(X)$, and hence $x \in V(I(X))$.

(b) "$\supset$": If $f \in \sqrt{J}$ then $f^k \in J$ for some $k \in \mathbb{N}$. It follows that $f^k(x) = 0$ for all $x \in V(J)$, hence also $f(x) = 0$ for all $x \in V(J)$, and thus $f \in I(V(J))$.

(a) "$\subset$": As X is an affine variety we know by Remark 1.6 that $X = V(J)$ for some ideal J. Then $I(V(J)) \supset \sqrt{J} \supset J$ by (b) "$\supset$", so taking the zero locus we obtain $V(I(V(J))) \subset V(J)$ by Lemma 1.4 (a). This means exactly that $V(I(X)) \subset X$.

Only the inclusion $I(V(J)) \subset \sqrt{J}$ of (b) "$\subset$" is hard; a proof of this result from commutative algebra can be found in [G3, Corollary 10.14]. It uses our Convention 1.1 (a) that K is algebraically closed.

The additional bijection statement now follows directly from (a) and (b), together with the observations that $\sqrt{J} = J$ since J is radical, $I(X)$ is actually a radical ideal by Remark 1.9 (b), and both
01 operations reverse inclusions by Lemma 1.4 (a) and Remark 1.9 (a). □

Example 1.11.

(a) Let $J \trianglelefteq K[x_1]$ be a non-zero ideal. As $K[x_1]$ is a principal ideal domain, we have $J = \langle f \rangle$ for a polynomial $f = (x_1 - a_1)^{k_1} \cdots (x_1 - a_r)^{k_r}$ for some distinct points $a_1,\dots,a_r \in \mathbb{A}^1$ and $k_1,\dots,k_r \in \mathbb{N}_{>0}$. The zero locus $V(J) = V(f) = \{a_1,\dots,a_r\} \subset \mathbb{A}^1$ then contains the data of the zeros of f, but no longer of the multiplicities $k_1,\dots,k_r$. Consequently, by Proposition 1.10

$$I(V(J)) = \sqrt{J} = \langle (x_1 - a_1) \cdots (x_1 - a_r) \rangle$$

is just the ideal of all polynomials vanishing at $a_1,\dots,a_r$ (with any order).

(b) If we had not assumed K to be algebraically closed, the Nullstellensatz would already break down in the simple example (a): The prime (and hence radical) ideal $J = \langle x_1^2 + 1 \rangle \trianglelefteq \mathbb{R}[x_1]$ has empty zero locus in $\mathbb{A}^1_{\mathbb{R}}$, so we would obtain $I(V(J)) = I(\emptyset) = \mathbb{R}[x_1] \neq J = \sqrt{J}$.

(c) The ideal $J = \langle x_1 - a_1, \ldots, x_n - a_n \rangle \trianglelefteq K[x_1, \ldots, x_n]$ is maximal (since $K[x_1, \ldots, x_n]/J \cong K$ is a field), and hence radical. As its zero locus is $V(J) = \{a\}$ for $a = (a_1, \ldots, a_n)$, we conclude by Proposition 1.10 that the ideal of the point a is

$$I(\{a\}) = I(V(J)) = J = \langle x_1 - a_1, \ldots, x_n - a_n \rangle.$$

In fact, points of $\mathbb{A}^n$ are clearly just the minimal non-empty varieties in $\mathbb{A}^n$, so by the inclusion-reversing operations of the Nullstellensatz they correspond exactly to the maximal (proper) ideals in $K[x_1, \ldots, x_n]$. Hence the bijection of Proposition 1.10 restricts to a bijection

$$\{\text{points in } \mathbb{A}^n\} \quad \stackrel{1:1}{\longleftrightarrow} \quad \{\text{maximal ideals in } K[x_1, \ldots, x_n]\},$$

so the maximal ideals considered above are actually the only maximal ideals in the polynomial ring.

First of all, Hilbert's Nullstellensatz of Proposition 1.10 allows us to translate the properties of $V(\,\cdot\,)$ in Lemma 1.7 to corresponding properties of the opposite operation $I(\,\cdot\,)$:

Lemma 1.12 (Properties of $I(\,\cdot\,)$)**.** *For any affine varieties X_1 and X_2 in $\mathbb{A}^n$ we have*

(a) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$;

(b) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof.

(a) A polynomial $f \in K[x_1, \ldots, x_n]$ is contained in $I(X_1 \cup X_2)$ if and only if $f(x) = 0$ for all $x \in X_1$ and all $x \in X_2$, which is the case if and only if $f \in I(X_1) \cap I(X_2)$.

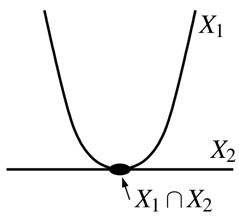
(b) By the Nullstellensatz of Proposition 1.10 we obtain

$$I(X_1 \cap X_2) \stackrel{1.10\,(a)}{=} I(V(I(X_1)) \cap V(I(X_2))) \stackrel{1.7\,(c)}{=} I(V(I(X_1) + I(X_2))) \stackrel{1.10\,(b)}{=} \sqrt{I(X_1) + I(X_2)}.$$

□

Remark 1.13. Recall from Remark 1.9 (b) that ideals of affine varieties are always radical. So in particular, Lemma 1.12 (a) shows that intersections of radical ideals are again radical — which could of course also be checked directly. In contrast, sums of radical ideals are in general not radical, which is why taking the radical in Lemma 1.12 (b) is really necessary.

There is also a geometric interpretation behind this fact. Consider for example the affine varieties $X_1, X_2 \subset \mathbb{A}^2_{\mathbb{C}}$ with ideals $I(X_1) = \langle x_2 - x_1^2 \rangle$ and $I(X_2) = \langle x_2 \rangle$ whose real points are shown on the right. Their intersection $X_1 \cap X_2$ is just the origin, with ideal $I(X_1 \cap X_2) = I(0) = \langle x_1, x_2 \rangle$ by Example 1.11 (c). But



$$I(X_1) + I(X_2) = \langle x_2 - x_1^2, x_2 \rangle = \langle x_1^2, x_2 \rangle$$

is not a radical ideal; only its radical is equal to $I(X_1 \cap X_2) = \langle x_1, x_2 \rangle$.

The geometric meaning of the non-radical ideal $I(X_1) + I(X_2) = \langle x_1^2, x_2 \rangle$ is simply that X_1 and X_2 are tangent at their intersection point: In a *linear* approximation their defining equations $x_2 = x_1^2$ and $x_2 = 0$ coincide and both describe the x_1-axis. Hence we could imagine that the intersection $X_1 \cap X_2$ extends from the origin to an infinitesimally small amount in the x_1-direction, as indicated in the picture — so that "x_1 does not quite vanish on the intersection (i. e. it does not lie in $I(X_1) + I(X_2)$), only x_1^2 does".

There are various ways to make this idea precise. In the "Plane Algebraic Curves" class, we would have said that X_1 and X_2 have intersection multiplicity $\dim_{\mathbb{C}} \mathbb{C}[x_1, x_2]/\langle x_1^2, x_2 \rangle = 2$ at the origin, encoding their tangential intersection numerically [G2, Chapter 2]. Here in these notes, we will instead introduce the notion of *schemes* in Chapter 12 that enlarges the geometric category of varieties to include "objects extending by infinitesimally small amounts in some directions", which will then yield a statement analogous to Proposition 1.10 that affine *subschemes* of $\mathbb{A}^n$ are in bijection to *arbitrary* ideals in $K[x_1,\dots,x_n]$. In this language, the intersection of X_1 and X_2 will then be the scheme corresponding to the non-radical ideal $\langle x_1^2, x_2 \rangle$ (see Construction 12.32).

Remark 1.14 (Consequences of Hilbert's Nullstellensatz)**.**

(a) (Weak Nullstellensatz) Let $J \trianglelefteq K[x_1,\dots,x_n]$ be an ideal in the polynomial ring. If $J \neq K[x_1,\dots,x_n]$ then J has a zero, i. e. $V(J)$ is non-empty: Otherwise we would have $\sqrt{J} = I(V(J)) = I(\emptyset) = K[x_1,\dots,x_n]$ by Proposition 1.10, which means $1 \in \sqrt{J}$ and gives us the contradiction $1 \in J$. This statement is usually called the **weak Nullstellensatz**; it can be thought of as a generalization of the algebraic closure property that a non-constant univariate polynomial has a zero. It also explains the origin of the name "Nullstellensatz" for Proposition 1.10.

(b) Another easy consequence of Proposition 1.10 is that polynomials and polynomial functions on $\mathbb{A}^n$ agree: If $f,g \in K[x_1,\dots,x_n]$ are two polynomials defining the same function on $\mathbb{A}^n$, i. e. such that $f(x) = g(x)$ for all $x \in \mathbb{A}^n$, then

$$f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{\langle 0 \rangle} = \langle 0 \rangle$$

and hence $f = g$ in $K[x_1,\dots,x_n]$. So $K[x_1,\dots,x_n]$ can be thought of as the ring of polynomial functions on $\mathbb{A}^n$.

It is easy to generalize this to an affine variety $X \subset \mathbb{A}^n$: Two polynomials $f,g \in K[x_1,\dots,x_n]$ define the same polynomial function on X, i. e. we have $f(x) = g(x)$ for all $x \in X$, if and only if $f - g \in I(X)$. So the quotient ring $K[x_1,\dots,x_n]/I(X)$ can be thought of as the ring of polynomial functions on X. Let us make this into a precise definition.

Definition 1.15 (Polynomial functions and coordinate rings)**.** Let $X \subset \mathbb{A}^n$ be an affine variety. A **polynomial function** on X is a map $X \to K$ that is of the form $x \mapsto f(x)$ for some $f \in K[x_1,\dots,x_n]$. By Remark 1.14 (b) the ring of all polynomial functions on X is just the quotient ring

$$A(X) := K[x_1,\dots,x_n]/I(X).$$

It is called the **coordinate ring** of the affine variety X.

Remark 1.16 (Coordinate rings are K-algebras)**.** Note that the coordinate ring $A(X)$ of an affine variety X is not just a ring, but also a *K-algebra* (i. e. it is also a K-vector space such that its ring multiplication is K-bilinear [G3, Definition 1.23 and Remark 1.24]). In fact, in the following we will often consider $A(X)$ as a K-algebra, despite its common name "coordinate ring" in the literature.

According to Definition 1.15, we can think of the elements of $A(X)$ both as functions on X and as elements of the quotient $K[x_1,\dots,x_n]/I(X)$. We can therefore use coordinate rings to define the concepts introduced so far in a relative setting, i. e. consider zero loci of ideals in $A(Y)$ and varieties contained in Y for a fixed ambient affine variety Y that is not necessarily $\mathbb{A}^n$.

Construction 1.17 (Relative version of $V(\cdot)$ and $I(\cdot)$)**.** Let $Y \subset \mathbb{A}^n$ be a fixed affine variety. The following two constructions are then completely analogous to those in Definitions 1.2 (b) and 1.8:

(a) For a subset $S \subset A(Y)$ of polynomial functions on Y we define its **zero locus** as

$$V(S) := V_Y(S) := \{x \in Y : f(x) = 0 \text{ for all } f \in S\} \quad \subset Y.$$

The subsets that are of this form are obviously precisely the affine varieties contained in Y. They are called **affine subvarieties** of Y.

(b) For a subset $X \subset Y$ the **ideal** of X in Y is defined to be

$$I(X) := I_Y(X) := \{f \in A(Y) : f(x) = 0 \text{ for all } x \in X\} \quad \trianglelefteq A(Y).$$

Remark 1.18. Let Y be an affine variety. With essentially the same arguments as before, all results considered in this chapter then hold in the relative setting of Construction 1.17 as well. Let us summarize them here again:

(a) In the same way as in Remark 1.14 (b) we see that $A(X) \cong A(Y)/I_Y(X)$ for any affine subvariety X of Y.

(b) (**Relative Nullstellensatz**) As in Proposition 1.10, we have $V_Y(I_Y(X)) = X$ for any affine subvariety X of Y and $I_Y(V_Y(J)) = \sqrt{J}$ for any ideal $J \trianglelefteq A(Y)$, giving rise to a bijection

$$\{\text{affine subvarieties of } Y\} \quad \stackrel{1:1}{\longleftrightarrow} \quad \{\text{radical ideals in } A(Y)\}.$$

A possible way to derive this from the absolute version is presented in Exercise 1.23.

(c) (Relative properties of $V(\cdot)$ and $I(\cdot)$) With the same proofs, the properties of $V(\cdot)$ of Lemma 1.7 and the properties of $I(\cdot)$ of Lemma 1.12 hold in the relative setting for ideals of $A(Y)$ resp. affine subvarieties of Y as well.

Exercise 1.19. Prove that every affine variety $X \subset \mathbb{A}^n$ consisting of only finitely many points can be written as the zero locus of n polynomials.

(Hint: Use interpolation. It is useful to assume first that all points in X have different x_1-coordinates.)

Exercise 1.20. Let X be an affine variety. Show that the coordinate ring $A(X)$ is a field if and only if X is a single point.

Exercise 1.21. Determine the radical of the ideal $\langle x_1^3 - x_2^6, x_1x_2 - x_2^3 \rangle \trianglelefteq \mathbb{C}[x_1,x_2]$.

(Hint: Hilbert's Nullstellensatz may be useful here.)

Exercise 1.22. Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes. Compute generators for the ideal $I(X)$, and show that $I(X)$ cannot be generated by fewer than three elements.

Exercise 1.23 (Relative Nullstellensatz, see Remark 1.18 (b))**.** Let $Y \subset \mathbb{A}^n$ be an affine variety, and denote by $\pi\colon K[x_1,\dots,x_n] \to K[x_1,\dots,x_n]/I(Y) = A(Y)$ the quotient map.

(a) Show that $V_Y(J) = V(\pi^{-1}(J))$ for every ideal J in $A(Y)$.

(b) Show that $\pi^{-1}(I_Y(X)) = I(X)$ for every affine subvariety X of Y.

(c) Use (a) and (b) to deduce the (interesting part of the) relative Nullstellensatz $I_Y(V_Y(J)) \subset \sqrt{J}$ for every ideal $J \trianglelefteq A(Y)$ from the corresponding absolute statement $I(V(J)) \subset \sqrt{J}$ for every ideal $J \trianglelefteq K[x_1,\dots,x_n]$ in Proposition 1.10. In particular, conclude that there is an inclusion-reversing bijection between affine subvarieties of Y and radical ideals in $A(Y)$.