7. Projective Varieties II: Ringed Spaces

After having defined projective varieties as topological spaces, we will now give them the structure of ringed spaces to make them into varieties in the sense of Chapter 5. In other words, we have to define a suitable notion of regular functions on (open subsets of) projective varieties.

Of course, as in the affine case in Definition 3.1 the general idea is that a regular function should be a K-valued function that is locally a quotient of two polynomials. However, note that in contrast to the affine situation the elements of the homogeneous coordinate ring $S(X)$ of a projective variety X are not well-defined functions on X: even if $f \in S(X)$ is homogeneous of degree d we only have $f(\lambda x) = \lambda^d f(x)$ for all $x \in X$ and $\lambda \in K$. So the only way to obtain well-defined functions is to consider quotients of homogeneous polynomials of the same degree, so that the factor λ^d cancels out:

Definition 7.1 (Regular functions on projective varieties)**.** Let U be an open subset of a projective variety X. A **regular function** on U is a map $\varphi : U \to K$ with the following property: for every $a \in U$ there are $d \in \mathbb{N}$ and $f, g \in S(X)_d$ with $f(x) \neq 0$ and

$$\varphi(x) = \frac{g(x)}{f(x)}$$

for all x in an open subset U_a with $a \in U_a \subset U$.

It is obvious that the sets $\mathscr{O}_X(U)$ of regular functions on U are subrings of the K-algebras of all functions from U to K, and — by the local nature of the definition — that they form a sheaf $\mathscr{O}_X$ on X.

With this definition, let us check first of all that the open subsets of a projective variety where one of the coordinates is non-zero are affine varieties, so that projective varieties are prevarieties in the sense of Definition 5.1.

Proposition 7.2 (Projective varieties are prevarieties)**.** *Let $X \subset \mathbb{P}^n$ be a projective variety. Then*

$$U_i = \{(x_0 : \cdots : x_n) \in X : x_i \neq 0\} \quad \subset X$$

is an affine variety for all $i = 0, \dots, n$. In particular, X is a prevariety.

Proof. By symmetry it suffices to prove the statement for $i = 0$. Let $X = V_p(h_1, \dots, h_r)$ for some homogeneous polynomials $h_1, \dots, h_r \in K[x_0, \dots, x_n]$, and set $g_j(x_1, \dots, x_n) = h_j(1, x_1, \dots, x_n)$ for all $j = 1, \dots, r$. If $Y = V_a(g_1, \dots, g_r)$ we claim that

$$F : Y \to U_0, \ (x_1, \dots, x_n) \mapsto (1 : x_1 : \cdots : x_n)$$

is an isomorphism with inverse

$$F^{-1} : U_0 \to Y, \ (x_0 : \cdots : x_n) \mapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right).$$

In fact, it is clear by construction that these two maps are well-defined and inverse to each other. Moreover, similarly to Remark 6.28 they are continuous: the inverse image of a closed set $V_p(f_1, \dots, f_s) \cap U_0$ under F is the closed set $V_a(f_1(1, \cdot), \dots, f_s(1, \cdot))$, and the image of a closed set $V_a(f_1, \dots, f_s) \subset Y$ under F is the closed set $V_p(f_1^h, \dots, f_s^h) \cap U_0$.

Finally, we have to check that F and F^{-1} pull back regular functions to regular functions: a regular function on (an open subset of) U_0 is by Definition 7.1 locally of the form $\frac{p(x_0, \dots, x_n)}{q(x_0, \dots, x_n)}$ (with nowhere vanishing denominator) for two homogeneous polynomials p and q of the same degree. Then

$$F^* \frac{p(x_0, \dots, x_n)}{q(x_0, \dots, x_n)} = \frac{p(1, x_1, \dots, x_n)}{q(1, x_1, \dots, x_n)}$$

is a quotient of polynomials and thus a regular function on Y. Conversely, F^{-1} pulls back a quotient $\frac{p(x_1,\dots,x_n)}{q(x_1,\dots,x_n)}$ of two polynomials to

$$(F^{-1})^* \frac{p(x_1,\dots,x_n)}{q(x_1,\dots,x_n)} = \frac{p\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)}{q\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)},$$

which is a regular function on U_0 since it can be rewritten as a quotient of two homogeneous polynomials of the same degree (by multiplying both the numerator and the denominator by x_0^m for $m = \max(\deg p, \deg q)$). Hence F is an isomorphism by Definition 4.3 (b), and so U_0 is an affine open subset of X.

In particular, as the open subsets U_i for $i = 0,\dots,n$ cover X we conclude that X is a prevariety. □

Exercise 7.3. Check that Definition 7.1 (together with Proposition 7.2) is compatible with our earlier constructions in the following cases:

(a) The prevariety $\mathbb{P}^1$ is the same as the one introduced in Example 5.5 (a).

(b) If $X \subset \mathbb{P}^n$ is a projective variety then its structure sheaf as defined above is the same as the closed subprevariety structure of X in $\mathbb{P}^n$ as in Construction 5.12 (b).

Exercise 7.4. Let $m,n \in \mathbb{N}_{>0}$. Use Exercise 6.32 to prove that $\mathbb{P}^m \times \mathbb{P}^n$ is not isomorphic to $\mathbb{P}^{m+n}$.

We have already mentioned that the major advantage of (subprevarieties of) projective varieties is that they have a global description with homogeneous coordinates that does not refer to gluing techniques. In fact, the following proposition shows that many morphisms between projective varieties can also be constructed without gluing.

Lemma 7.5 (Morphisms of projective varieties)**.** *Let $X \subset \mathbb{P}^n$ be a projective variety, and let $f_0,\dots,f_m \in S(X)$ be homogeneous elements of the same degree. Then on the open subset $U := X\backslash V(f_0,\dots,f_m)$ these elements define a morphism*

$$f : U \to \mathbb{P}^m,\ \ x \mapsto (f_0(x):\dots:f_m(x)).$$

Proof. First of all note that f is well-defined set-theoretically: by definition of U the image point can never be $(0:\dots:0)$; and if we rescale the homogeneous coordinates $x_0,\dots,x_n$ of $x \in U$ we get

$$\begin{aligned}(f_0(\lambda x_0:\dots:\lambda x_n):\dots:f_m(\lambda x_0:\dots:\lambda x_n))\\ &= (\lambda^d f_0(x_0:\dots:x_n):\dots:\lambda^d f_m(x_0:\dots:x_n))\\ &= (f_0(x_0:\dots:x_n):\dots:f_m(x_0:\dots:x_n)),\end{aligned}$$

where d is the common degree of the $f_0,\dots,f_m$. To check that f is a morphism we want to use the gluing property of Lemma 4.6. So let $\{V_i : i = 0,\dots,m\}$ be the affine open cover of $\mathbb{P}^m$ with $V_i = \{(y_0:\dots:y_m) : y_i \neq 0\}$ for all i. Then the open subsets $U_i := f^{-1}(V_i) = \{x \in X : f_i(x) \neq 0\}$ cover U, and in the affine coordinates on V_i the map $f|_{U_i}$ is given by the quotients of polynomials $\frac{f_j}{f_i}$ for $j = 0,\dots,m$ with $j \neq i$, which are regular functions on U_i by Definition 7.1. Hence $f|_{U_i}$ is a morphism by Proposition 4.7, and so f is a morphism by Lemma 4.6. □

Example 7.6.

(a) Let $A \in \mathrm{GL}(n+1,K)$ be an invertible matrix. Then $f : \mathbb{P}^n \to \mathbb{P}^n$, $x \mapsto Ax$ is a morphism with inverse $f^{-1} : \mathbb{P}^n \to \mathbb{P}^n$, $x \mapsto A^{-1}x$, and hence an isomorphism. We will refer to these maps as **projective automorphisms** of $\mathbb{P}^n$. In fact, we will see in Proposition 13.4 that these are the only isomorphisms of $\mathbb{P}^n$.

(b) Let $a = (1:0:\dots:0) \in \mathbb{P}^n$ and $L = V(x_0) \cong \mathbb{P}^{n-1}$. Then the map

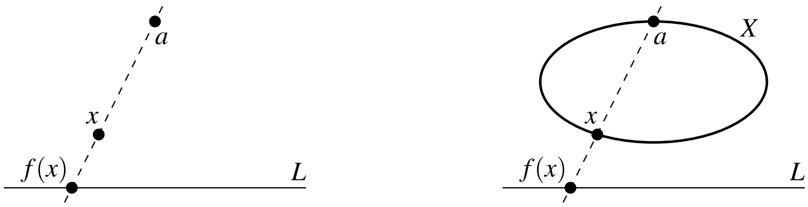
$$f : \mathbb{P}^n\backslash\{a\} \to \mathbb{P}^{n-1},\ \ (x_0:\dots:x_n) \mapsto (x_1:\dots:x_n)$$

given by forgetting one of the homogeneous coordinates is a morphism by Lemma 7.5. It can be interpreted geometrically as in the picture below on the left: for $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n \backslash \{a\}$ the unique line through a and x is clearly given parametrically by

$$\{(s : t\,x_1 : \cdots : t\,x_n) : (s:t) \in \mathbb{P}^1\},$$

and its intersection point with L is just $(0 : x_1 : \cdots : x_n)$, i. e. $f(x)$ with the identification $L \cong \mathbb{P}^{n-1}$. We call f the **projection** from a to the linear subspace L. Note however that the picture below is only schematic and does not show a standard affine open subset $U_i = \{(x_0 : \cdots : x_n) : x_i \neq 0\}$, since none of these subsets contains both a and (a non-empty open subset of) L.

Of course, the same construction works for any point $a \in \mathbb{P}^n$ and any linear subspace L of dimension $n-1$ not containing a — the corresponding morphism then differs from the special one considered above by a projective automorphism as in (a).



(b) (c)

(c) The projection morphism $f : \mathbb{P}^n \backslash \{a\} \to \mathbb{P}^{n-1}$ as in (b) cannot be extended to the point a. The intuitive reason for this is that the line through a and x (and thus also the point $f(x)$) does not have a well-defined limit as x approaches a. This changes however if we restrict the projection to a suitable projective variety: for $X = V(x_0x_2 - x_1^2)$ as in the schematic picture above on the right consider the map

$$f : X \to \mathbb{P}^1, \ \ (x_0 : x_1 : x_2) \mapsto \begin{cases} (x_1 : x_2) & \text{if } (x_0 : x_1 : x_2) \neq (1:0:0), \\ (x_0 : x_1) & \text{if } (x_0 : x_1 : x_2) \neq (0:0:1). \end{cases}$$

It is clearly well-defined since the equation $x_0x_2 - x_1^2 = 0$ implies $(x_1 : x_2) = (x_0 : x_1)$ whenever both these points in $\mathbb{P}^1$ are defined. Moreover, it extends the projection as in (b) to all of X (which includes the point a), and it is a morphism since it is patched together from two projections as above. Geometrically, the image $f(a)$ is the intersection of the tangent to X at a with the line L.

This geometric picture also tells us that f is bijective: for every point $y \in L$ the restriction of the polynomial $x_0x_2 - x_1^2$ defining X to the line through a and y has degree 2, and thus this line intersects X in two points (counted with multiplicities), of which one is a. The other point is then the unique inverse image $f^{-1}(y)$. In fact, it is easy to check that f is even an isomorphism since its inverse is

$$f^{-1} : \mathbb{P}^1 \to X, \ \ (y_0 : y_1) \mapsto (y_0^2 : y_0y_1 : y_1^2),$$

which is a morphism by Lemma 7.5.

Note that the example of the morphism f above also shows that we cannot expect every morphism between projective varieties to have a global description by homogeneous polynomials as in Lemma 7.5.

(d) Now let $X \subset \mathbb{P}^2$ be any *projective conic*, i. e. the zero locus of a single irreducible homogeneous polynomial $f \in K[x_0, x_1, x_2]$ of degree 2. Assuming that $\operatorname{char} K \neq 2$, we know by Exercise 4.12 that the affine part $X \cap \mathbb{A}^2$ is isomorphic to $V_a(x_2 - x_1^2)$ or $V_a(x_1x_2 - 1)$ by a linear transformation followed by a translation. Extending this map to a projective automorphism of $\mathbb{P}^2$ as in (a), the projective conic X thus becomes isomorphic to $V_p(x_0x_2 - x_1^2)$ or $V_p(x_1x_2 - x_0^2)$ by Proposition 6.33 (b). So by (c) we see that every projective conic is isomorphic to $\mathbb{P}^1$.

Exercise 7.7. Let us say that $n+2$ points in $\mathbb{P}^n$ are *in general position* if for any $n+1$ of them their representatives in K^{n+1} are linearly independent.

Now let $a_1,\dots,a_{n+2}$ and $b_1,\dots,b_{n+2}$ be two sets of points in $\mathbb{P}^n$ in general position. Show that there is an isomorphism $f:\mathbb{P}^n\to\mathbb{P}^n$ with $f(a_i)=b_i$ for all $i=1,\dots,n+2$.

Exercise 7.8. Show by example that the homogeneous coordinate ring of a projective variety is *not* invariant under isomorphisms, i.e. that there are isomorphic projective varieties X,Y such that the rings $S(X)$ and $S(Y)$ are not isomorphic.

Exercise 7.9. Let $f:\mathbb{P}^n\to\mathbb{P}^m$ be a morphism. Prove:

(a) If $X\subset\mathbb{P}^m$ is the zero locus of a single homogeneous polynomial in $K[x_0,\dots,x_m]$ then every irreducible component of $f^{-1}(X)$ has dimension at least $n-1$.

(b) If $n>m$ then f must be constant.

Let us now verify that projective varieties are separated, i.e. that they are varieties and not just prevarieties. In other words, we have to check that the diagonal Δ_X of a projective variety X is closed in the product $X\times X$. By Lemma 5.20 (b) it suffices to show this for $X=\mathbb{P}^n$.

For the proof of this statement it is useful to first find a good description of the product of projective spaces — note that by Exercise 7.4 such products are not just again projective spaces. Of course, we could just parametrize these products by two sets of homogeneous coordinates. It turns out however that we can also use a single set of homogeneous coordinates and thus embed products of projective spaces as a projective variety into a bigger projective space.

Construction 7.10 (Segre embedding)**.** Consider $\mathbb{P}^m$ with homogeneous coordinates $x_0,\dots,x_m$ and $\mathbb{P}^n$ with homogeneous coordinates $y_0,\dots,y_n$. Set $N=(m+1)(n+1)-1$ and let $\mathbb{P}^N$ be the projective space with homogeneous coordinates labeled $z_{i,j}$ for $0\le i\le m$ and $0\le j\le n$. Then there is an obviously well-defined set-theoretic map

$$f:\mathbb{P}^m\times\mathbb{P}^n\to\mathbb{P}^N$$

11 given by $z_{i,j}=x_i\,y_j$ for all i,j. It satisfies the following properties:

Proposition 7.11. *Let $f:\mathbb{P}^m\times\mathbb{P}^n\to\mathbb{P}^N$ be the map of Construction 7.10. Then:*

(a) *The image $X=f(\mathbb{P}^m\times\mathbb{P}^n)$ is a projective variety given by*

$$X=V_p(z_{i,j}\,z_{k,l}-z_{i,l}\,z_{k,j}:0\le i,k\le m, 0\le j,l\le n).$$

(b) *The map $f:\mathbb{P}^m\times\mathbb{P}^n\to X$ is an isomorphism.*

In particular, $\mathbb{P}^m\times\mathbb{P}^n\cong X$ is a projective variety. The isomorphism $f:\mathbb{P}^m\times\mathbb{P}^n\to X\subset\mathbb{P}^N$ is called the ***Segre embedding****; the coordinates $z_{0,0},\dots,z_{m,n}$ above will be referred to as* ***Segre coordinates*** *on $\mathbb{P}^m\times\mathbb{P}^n$.*

Proof.

(a) It is obvious that the points of $f(\mathbb{P}^m\times\mathbb{P}^n)$ satisfy the given equations. Conversely, consider a point $z\in\mathbb{P}^N$ with homogeneous coordinates $z_{0,0},\dots,z_{m,n}$ that satisfy the given equations. At least one of these coordinates must be non-zero; we can assume without loss of generality that it is $z_{0,0}$. Let us pass to affine coordinates by setting $z_{0,0}=1$. Then we have $z_{i,j}=z_{i,0}z_{0,j}$ for all $i=0,\dots,m$ and $j=0,\dots,n$. Hence by setting $x_i=z_{i,0}$ and $y_j=z_{0,j}$ (in particular $x_0=y_0=1$) we obtain a point of $\mathbb{P}^m\times\mathbb{P}^n$ that is mapped to z by f.

(b) Continuing the above notation, let $z\in X$ be a point with $z_{0,0}=1$. If $f(x,y)=z$ for some $(x,y)\in\mathbb{P}^m\times\mathbb{P}^n$, it follows that $x_0\neq 0$ and $y_0\neq 0$, so we can pass to affine coordinates here as well and assume that $x_0=1$ and $y_0=1$. But then it follows that $x_i=z_{i,0}$ and $y_j=z_{0,j}$ for all i and j, i.e. f is injective and thus as a map onto its image also bijective.

The same calculation shows that f and f^{-1} are given (locally in affine coordinates) by polynomial maps. Hence f is an isomorphism. $\square$

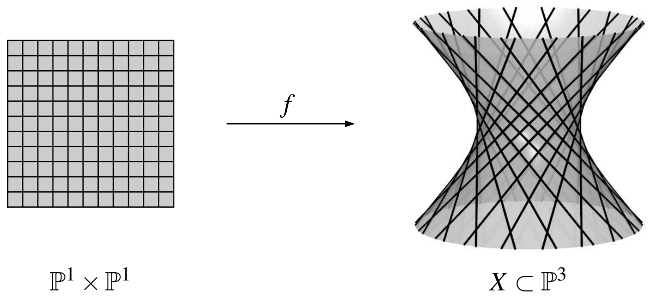
Example 7.12. According to Proposition 7.11, the product $\mathbb{P}^1 \times \mathbb{P}^1$ is (isomorphic to) the surface

$$X = \{(z_{0,0} : z_{0,1} : z_{1,0} : z_{1,1}) : z_{0,0}\, z_{1,1} = z_{1,0}\, z_{0,1}\} \quad \subset \mathbb{P}^3$$

by the isomorphism

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \to X, \;\; ((x_0 : x_1), (y_0 : y_1)) \mapsto (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1).$$

In particular, the "lines" $\{a\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{a\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ where the first or second factor is constant, respectively, are mapped to lines in $X \subset \mathbb{P}^3$. The following schematic picture shows these two families of lines on the surface X (whose set of real points is a hyperboloid).



Corollary 7.13. *Every projective variety is a variety.*

Proof. We have already seen in proposition 7.2 that every projective variety is a prevariety. So by Lemma 5.20 (b) it only remains to be shown that $\mathbb{P}^n$ is separated, i. e. that the diagonal $\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$. We can describe this diagonal as

$$\Delta_{\mathbb{P}^n} = \{((x_0 : \cdots : x_n), (y_0 : \cdots : y_n)) : x_i y_j - x_j y_i = 0 \text{ for all } i, j\},$$

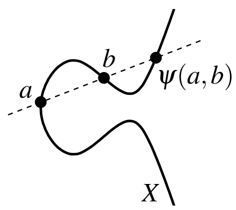
because these equations mean exactly that the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n \end{pmatrix}$$

has rank (at most) 1, i. e. that $(x_0 : \cdots : x_n) = (y_0 : \cdots : y_n)$. In particular, it follows that $\Delta_{\mathbb{P}^n}$ is closed as the zero locus of the homogeneous linear polynomials $z_{i,j} - z_{j,i}$ in the Segre coordinates $z_{i,j} = x_i y_j$ of $\mathbb{P}^n \times \mathbb{P}^n$. □

Remark 7.14. If $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are projective varieties then $X \times Y$ is a closed subset of $\mathbb{P}^m \times \mathbb{P}^n$. As the latter is a projective variety by the Segre embedding we see that $X \times Y$ is a projective variety as well (namely a projective subvariety of $\mathbb{P}^m \times \mathbb{P}^n$).

Exercise 7.15. Let $X \subset \mathbb{P}^2$ be a curve given as the zero locus of a homogeneous polynomial of degree 3. Moreover, let $U \subset X \times X$ be the set of all $(a, b) \in X \times X$ such that $a \neq b$ and the unique line through a and b meets X in exactly three distinct points. Of course, two of these points are then a and b; we will denote the third one by $\psi(a, b) \in X$.



Show that $U \subset X \times X$ is open, and that $\psi : U \to X$ is a morphism.

Exercise 7.16.

(a) Prove that for every projective variety $Y \subset \mathbb{P}^n$ of pure dimension $n-1$ there is a homogeneous polynomial f such that $I(Y) = (f)$. You may use the commutative algebra fact that every polynomial in $K[x_0, \dots, x_n]$ admits a unique decomposition into prime elements [G5, Remark 8.6].

(b) If X is a projective variety of dimension n, show by example that in general not every projective variety $Y \subset X$ of dimension $n-1$ is of the form $V(f)$ for a homogeneous polynomial $f \in S(X)$. (One possibility is to consider the Segre embedding X of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, and $Y = \mathbb{P}^1 \times \{0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$.)

The most important property of projective varieties is that they are compact in the classical topology if the ground field is $\mathbb{C}$. We have seen this already for projective spaces in Remark 6.4, and it then follows for projective varieties as well since they are closed subsets of them. However, Exercises 2.34 (c) and 5.11 (a) show unfortunately that every prevariety is compact in the Zariski topology, and so in particular that compactness in the Zariski topology does not capture the same geometric idea as in the classical case. We therefore need an alternative description of the intuitive compactness property that works in our algebraic setting of the Zariski topology.

The key idea to achieve this is that compact sets should be mapped to compact sets again under continuous maps. In our language, this means that images of morphisms between projective varieties should be closed. This property (that we have already seen to be false for general varieties in Remark 5.15 (a)) is what we want to prove now. We start with a special case which contains all the hard work, and from which the general case will then follow easily.

Definition 7.17 (Closed maps)**.** A map $f : X \to Y$ between topological spaces is called **closed** if $f(A) \subset Y$ is closed for every closed subset $A \subset X$.

Proposition 7.18. *The projection map $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed.*

Proof. Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ be a closed set. By Remark 6.20 we can write $Z = V(f_1,\dots,f_r)$ for homogeneous polynomials $f_1,\dots,f_r$ of the same degree d in the Segre coordinates of $\mathbb{P}^n \times \mathbb{P}^m$, i. e. for bihomogeneous polynomials of degree d in both the coordinates $x_0,\dots,x_n$ of $\mathbb{P}^n$ and $y_0,\dots,y_m$ of $\mathbb{P}^m$. Now consider a fixed point $a \in \mathbb{P}^m$; we will determine if it is contained in the image $\pi(Z)$. To do this, let $g_i = f_i(\,\cdot\,,a) \in K[x_0,\dots,x_n]$ for $i = 1,\dots,r$. Then

$$\begin{aligned}
a \notin \pi(Z) &\Leftrightarrow \text{there is no } x \in \mathbb{P}^n \text{ with } (x,a) \in Z \\
&\Leftrightarrow V_p(g_1,\dots,g_r) = \emptyset \\
&\Leftrightarrow \sqrt{(g_1,\dots,g_r)} = (1) \text{ or } \sqrt{(g_1,\dots,g_r)} = (x_0,\dots,x_n) \quad \text{(Proposition 6.22)} \\
&\Leftrightarrow \text{there are } k_i \in \mathbb{N} \text{ with } x_i^{k_i} \in (g_1,\dots,g_r) \text{ for all } i \\
&\Leftrightarrow K[x_0,\dots,x_n]_k \subset (g_1,\dots,g_r) \text{ for some } k \in \mathbb{N},
\end{aligned}$$

where as usual $K[x_0,\dots,x_n]_k$ denotes the homogeneous degree-k part of the polynomial ring as in Definition 6.6, and the direction "$\Rightarrow$" of the last equivalence follows by setting $k = k_0 + \cdots + k_n$. Of course, the last condition can only be satisfied if $k \geq d$ and is equivalent to $K[x_0,\dots,x_n]_k = (g_1,\dots,g_r)_k$. As $(g_1,\dots,g_r) = \{h_1g_1 + \cdots + h_rg_r : h_1,\dots,h_r \in K[x_0,\dots,x_n]\}$ this is the same as saying that the K-linear map

$$F_k : (K[x_0,\dots,x_n]_{k-d})^r \to K[x_0,\dots,x_n]_k, \;\; (h_1,\dots,h_r) \mapsto h_1g_1 + \cdots h_rg_r$$

is surjective, i. e. has rank $\dim_K K[x_0,\dots,x_n]_k = \binom{n+k}{k}$ for some $k \geq d$. This in turn is the case if and only if at least one of the minors of size $\binom{n+k}{k}$ of a matrix for some F_k is non-zero. But these minors are polynomials in the coefficients of g and thus in the coordinates of a, and consequently the non-vanishing of one of these minors is an open condition in the Zariski topology of $\mathbb{P}^m$.

Hence the set of all $a \in \mathbb{P}^m$ with $a \notin \pi(Z)$ is open, which means that $\pi(Z)$ is closed. □

Remark 7.19. Let us look at Proposition 7.18 from an algebraic point of view. We start with some equations $f_1(x,y) = \cdots = f_r(x,y) = 0$ in two sets of variables $x = (x_0,\dots,x_n)$ and $y = (y_0,\dots,y_m)$ and ask for the image of their common zero locus under the projection map $(x,y) \mapsto x$. The equations satisfied on this image are precisely the equations in x alone that can be derived from the given ones $f_1(x,y) = \cdots = f_r(x,y) = 0$ in x and y. In other words, we want to *eliminate* the variables y from the given system of equations. The statement of Proposition 7.18 is therefore sometimes called the *main theorem of elimination theory*.

Corollary 7.20. *The projection map $\pi : \mathbb{P}^n \times Y \to Y$ is closed for any variety Y.*

Proof. Let us first show the statement for an affine variety $Y \subset \mathbb{A}^m$. Then we can regard Y as a locally closed subvariety of $\mathbb{P}^m$ via the embedding $\mathbb{A}^m \subset \mathbb{P}^m$. Now let $Z \subset \mathbb{P}^n \times Y$ be closed, and let

$\overline{Z}$ be its closure in $\mathbb{P}^n \times \mathbb{P}^m$. If $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is the projection map then $\pi(\overline{Z})$ is closed in $\mathbb{P}^m$ by Proposition 7.18, and thus

$$\pi(Z) = \pi(\overline{Z} \cap (\mathbb{P}^n \times Y)) = \pi(\overline{Z}) \cap Y$$

is closed in Y.

If Y is any variety we can cover it by affine open subsets. As the condition that a subset is closed can be checked by restricting it to the elements of an open cover, the statement follows from the corresponding one for the affine open patches that we have just shown. □

It is in fact this property of Corollary 7.20 that captures the classical idea of compactness. Let us therefore give it a name:

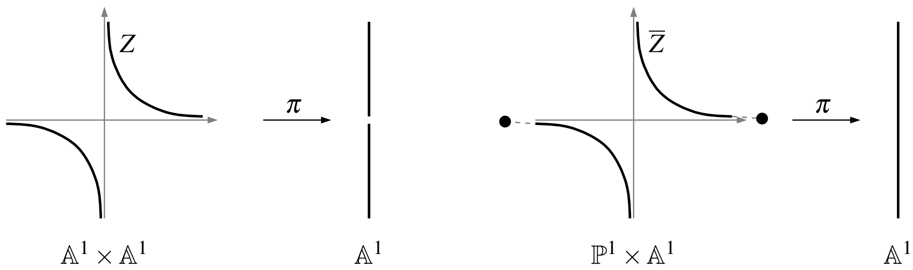
Definition 7.21 (Complete varieties)**.** A variety X is called **complete** if the projection map $\pi : X \times Y \to Y$ is closed for any variety Y.

Example 7.22.

(a) $\mathbb{P}^n$ is complete by Corollary 7.20.

(b) Any closed subvariety X' of a complete variety X is complete: if $Z \subset X' \times Y$ is closed then Z is also closed in $X \times Y$, and hence its image under the second projection to Y is closed as well. In particular, by (a) this means that every projective variety is complete.

(c) $\mathbb{A}^1$ is not complete: as in the picture below on the left, the image $\pi(Z)$ of the closed subset $Z = V(x_1 x_2 - 1) \subset \mathbb{A}^1 \times \mathbb{A}^1$ under the second projection is $\mathbb{A}^1 \backslash \{0\}$, which is not closed.



The geometric reason for this is that $\mathbb{A}^1$ is missing a point at infinity: if we replace $\mathbb{A}^1$ by $\mathbb{P}^1$ as in the picture on the right there is an additional point in the closure $\overline{Z}$ of $Z \subset \mathbb{A}^1 \times \mathbb{A}^1$ in $\mathbb{P}^1 \times \mathbb{A}^1$; the image of this point under π fills the gap and makes $\pi(\overline{Z})$ a closed set. Intuitively, one can think of the name "complete" as coming from the geometric idea that it contains all the "points at infinity" that are missing in affine varieties.

Remark 7.23. There are complete varieties that are not projective, but this is actually quite hard to show — we will certainly not meet such an example in this course. So for practical purposes you can usually assume that the terms "projective variety" and "complete variety" are synonymous.

In any case, complete varieties now have the property that we were aiming for:

Corollary 7.24. *Let $f : X \to Y$ be a morphism of varieties. If X is complete then its image $f(X)$ is closed in Y.*

Proof. By Proposition 5.21 (a) the graph $\Gamma_f \subset X \times Y$ is closed. But then $f(X) = \pi(\Gamma_f)$ for the projection map $\pi : X \times Y \to Y$, which is closed again since X is complete. □

Let us conclude this chapter with two applications of this property.

Corollary 7.25. *Let X be a connected complete variety. Then $\mathscr{O}_X(X) = K$, i. e. every global regular function on X is constant.*

Proof. A global regular function $\varphi \in \mathscr{O}_X(X)$ determines a morphism $\varphi : X \to \mathbb{A}^1$. By extension of the target we can consider this as a morphism $\varphi : X \to \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ whose image $\varphi(X) \subset \mathbb{P}^1$ does not contain the point ∞. But $\varphi(X)$ is also closed by Corollary 7.24 since X is complete, and hence it must be a finite set since these are the only closed proper subsets of $\mathbb{P}^1$. Moreover, Exercise 2.21 (b) implies that $\varphi(X)$ is connected since X is. Altogether this means that $\varphi(X)$ is a single point, i. e. that φ is constant. □

Remark 7.26. Corollary 7.25 is another instance of a result that has a counterpart in complex analysis: it can be shown that every holomorphic function on a connected compact complex manifold is constant.

12

Construction 7.27 (Veronese embedding)**.** Choose $n,d \in \mathbb{N}_{>0}$, and let $f_0,\dots,f_N \in K[x_0,\dots,x_n]$ for $N = \binom{n+d}{n} - 1$ be the set of all monomials of degree d in the variables $x_0,\dots,x_n$, in any order. Consider the map

$$F : \mathbb{P}^n \to \mathbb{P}^N, \ x \mapsto (f_0(x) : \cdots : f_N(x)).$$

By Lemma 7.5 this is a morphism (note that the monomials $x_0^d,\dots,x_n^d$, which cannot be simultaneously zero, are among the $f_0,\dots,f_N$). So by Corollary 7.24 the image $X = F(\mathbb{P}^n)$ is a projective variety.

We claim that $F : \mathbb{P}^n \to X$ is an isomorphism. All we have to do to prove this is to find an inverse morphism. This is not hard: we can do this on an affine open cover, so let us e. g. consider the open subset where $x_0 \neq 0$, i. e. $x_0^d \neq 0$. On this set we can pass to affine coordinates and set $x_0 = 1$. The inverse morphism is then given by $x_i = \frac{x_i x_0^{d-1}}{x_0^d}$ for $i = 1,\dots,n$, which is a quotient of two degree-d monomials.

The morphism F is therefore an isomorphism and thus realizes $\mathbb{P}^n$ as a subvariety X of $\mathbb{P}^N$. It is usually called the degree-d **Veronese embedding**; the coordinates on $\mathbb{P}^N$ are called **Veronese coordinates** of $\mathbb{P}^n \cong X$. Of course, this embedding can also be restricted to any projective variety $Y \subset \mathbb{P}^n$ and then gives an isomorphism by degree-d polynomials between Y and a projective variety in $\mathbb{P}^N$.

The importance of the Veronese embedding lies in the fact that degree-d polynomials in the coordinates of $\mathbb{P}^n$ are translated into *linear* polynomials in the Veronese coordinates. An example where this is useful will be given in Corollary 7.30.

Example 7.28.

(a) For $d = 1$ the Veronese embedding of $\mathbb{P}^n$ is just the identity $\mathbb{P}^n \to \mathbb{P}^n$.

(b) For $n = 1$ the degree-d Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^d$ is

$$F : \mathbb{P}^1 \to \mathbb{P}^d, \ (x_0 : x_1) \mapsto (x_0^d : x_0^{d-1}x_1 : \cdots : x_0 x_1^{d-1} : x_1^d).$$

In the $d = 2$ case we have already seen in Example 7.6 (c) that this is an isomorphism.

Exercise 7.29. Let $F : \mathbb{P}^n \to \mathbb{P}^N$ be the degree-d Veronese embedding as in Construction 7.27, with $N = \binom{n+d}{n} - 1$. By applying Corollary 7.24 we have seen already that the image $X = F(\mathbb{P}^n)$ is a projective variety. Find explicit equations describing X, i. e. generators for a homogeneous ideal I such that $X = V(I)$.

Corollary 7.30. *Let $X \subset \mathbb{P}^n$ be a projective variety, and let $f \in S(X)$ be a non-zero homogeneous element. Then $X \backslash V(f)$ is an affine variety.*

Proof. If $f = x_0$ this is just Proposition 7.2. For a general linear polynomial f the statement follows from this after a projective automorphism as in Example 7.6 (a) that takes f to x_0, and if f is of degree $d > 1$ we can reduce the claim to the linear case by first applying the degree-d Veronese embedding of Construction 7.27. □

Exercise 7.31. Recall from Example 7.6 (d) that a conic in $\mathbb{P}^2$ over a field of characteristic not equal to 2 is the zero locus of an irreducible homogeneous polynomial of degree 2 in $K[x_0,x_1,x_2]$.

(a) Considering the coefficients of such polynomials, show that the set of all conics in $\mathbb{P}^2$ can be identified with an open subset U of the projective space $\mathbb{P}^5$.

(b) Let $a \in \mathbb{P}^2$. Show that the subset of U consisting of all conics passing through a is the zero locus of a linear equation in the homogeneous coordinates of $U \subset \mathbb{P}^5$.

(c) Given 5 points in $\mathbb{P}^2$, no three of which lie on a line, show that there is a unique conic in $\mathbb{P}^5$ passing through all these points.

Exercise 7.32. Let $X \subset \mathbb{P}^3$ be the degree-3 Veronese embedding of $\mathbb{P}^1$, i. e. the image of the morphism

$$\mathbb{P}^1 \to \mathbb{P}^3, \ (x_0:x_1) \mapsto (y_0:y_1:y_2:y_3) = (x_0^3:x_0^2x_1:x_0x_1^2:x_1^3).$$

Moreover, let $a = (0:0:1:0) \in \mathbb{P}^3$ and $L = V(y_2) \subset \mathbb{P}^3$, and consider the projection f from a to L as in Example 7.6 (b).

(a) Show that f is a morphism.

(b) Determine an equation of the curve $f(X)$ in $L \cong \mathbb{P}^2$.

(c) Is $f : X \to f(X)$ an isomorphism onto its image?