6. Projective Varieties I: Topology

In the last chapter we have studied (pre-)varieties, i. e. topological spaces that are locally isomorphic to affine varieties. In particular, the ability to glue affine varieties together allowed us to construct *compact* spaces (in the classical topology over the ground field $\mathbb{C}$) as e. g. $\mathbb{P}^1$, whereas affine varieties themselves are never compact unless they consist of only finitely many points (see Exercise 2.34 (c)). Unfortunately, the description of a variety in terms of its affine patches and gluing isomorphisms is quite inconvenient in practice, as we have seen already in the calculations in the last chapter. It would therefore be desirable to have a global description of these spaces that does not refer to gluing methods.

Projective varieties form a very large class of "compact" varieties that do admit such a global description. In fact, the class of projective varieties is so large that it is not easy to construct a variety that is *not* (an open subset of) a projective variety — in this class we will certainly not see one.

In this chapter we will construct projective varieties as topological spaces, leaving their structure as ringed spaces to Chapter 7. To do this we first of all need projective spaces, which can be thought of as compactifications of affine spaces. We have already seen $\mathbb{P}^1$ as $\mathbb{A}^1$ together with a "point at infinity" in Example 5.5 (a); other projective spaces are just generalizations of this construction to higher dimensions. As we aim for a global description of these spaces however, their definition looks quite different from the one in Example 5.5 (a) at first.

Definition 6.1 (Projective spaces)**.** Let $n \in \mathbb{N}$. We define **projective n-space** over K, denoted $\mathbb{P}^n_K$ or simply $\mathbb{P}^n$, to be the set of all 1-dimensional linear subspaces of the vector space K^{n+1}.

Notation 6.2 (Homogeneous coordinates)**.** Obviously, a 1-dimensional linear subspace of K^{n+1} is uniquely determined by a non-zero vector in K^{n+1}, with two such vectors spanning the same linear subspace if and only if they are scalar multiples of each other. In other words, we have

$$\mathbb{P}^n = (K^{n+1}\backslash\{0\}) / \sim$$

with the equivalence relation

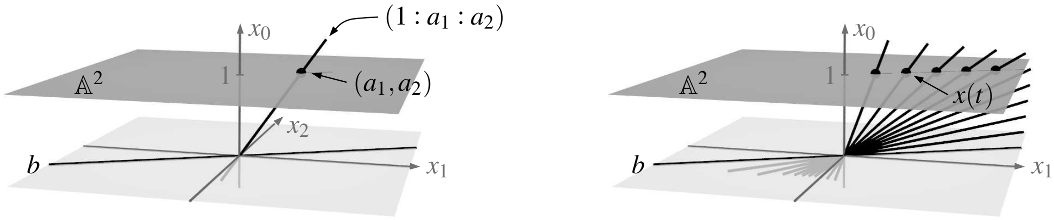
$$(x_0,\dots,x_n) \sim (y_0,\dots,y_n) \quad :\Leftrightarrow \quad x_i = \lambda y_i \text{ for some } \lambda \in K^* \text{ and all } i,$$

where $K^* = K\backslash\{0\}$ is the multiplicative group of units of K. This is usually written as $\mathbb{P}^n = (K^{n+1}\backslash\{0\})/K^*$, and the equivalence class of $(x_0,\dots,x_n)$ will be denoted by $(x_0:\dots:x_n) \in \mathbb{P}^n$ (the notations $[x_0:\dots:x_n]$ and $[x_0,\dots,x_n]$ are also common in the literature). So in the notation $(x_0:\dots:x_n)$ for a point in $\mathbb{P}^n$ the numbers $x_0,\dots,x_n$ are not all zero, and they are defined only up to a common scalar multiple. They are called the **homogeneous coordinates** of the point (the reason for this name will become obvious in the course of this chapter). Note also that we will usually label the homogeneous coordinates of $\mathbb{P}^n$ by $x_0,\dots,x_n$ instead of by $x_1,\dots,x_{n+1}$. This choice is motivated by the following relation between $\mathbb{A}^n$ and $\mathbb{P}^n$.

Remark 6.3 (Geometric interpretation of $\mathbb{P}^n$)**.** Consider the map

$$f : \mathbb{A}^n \to \mathbb{P}^n,\ (x_1,\dots,x_n) \mapsto (1:x_1:\dots:x_n).$$

As in the picture below on the left we can embed the affine space $\mathbb{A}^n$ in K^{n+1} at the height $x_0 = 1$, and then think of f as mapping a point to the 1-dimensional linear subspace spanned by it.



The map f is obviously injective, with image $U_0 := \{(x_0 : \cdots : x_n) : x_0 \neq 0\}$. On this image the inverse of f is given by

$$f^{-1} : U_0 \to \mathbb{A}^n, \; (x_0 : \cdots : x_n) \mapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right),$$

it sends a line through the origin to its intersection point with $\mathbb{A}^n$ embedded in K^{n+1}. We can thus think of $\mathbb{A}^n$ as a subset U_0 of $\mathbb{P}^n$. The coordinates $\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$ of a point $(x_0 : \cdots : x_n) \in U_0 \subset \mathbb{P}^n$ are called its **affine coordinates**.

The remaining points of $\mathbb{P}^n$ are of the form $(0 : x_1 : \cdots : x_n)$; in the picture above they correspond to lines in the horizontal plane through the origin, such as e. g. b. By forgetting their first coordinate (which is zero anyway) they form a set that is naturally bijective to $\mathbb{P}^{n-1}$. Thinking of the ground field $\mathbb{C}$ we can regard them as *points at infinity*: consider e. g. in $\mathbb{A}^n_{\mathbb{C}}$ a parametrized line

$$x(t) = (a_1 + b_1 t, \ldots, a_n + b_n t) \quad \text{for } t \in \mathbb{C}$$

for some starting point $(a_1, \ldots, a_n) \in \mathbb{C}^n$ and direction vector $(b_1, \ldots, b_n) \in \mathbb{C}^n \backslash \{0\}$. Of course, there is no limit point of $x(t)$ in $\mathbb{A}^n_{\mathbb{C}}$ as $t \to \infty$. But if we embed $\mathbb{A}^n_{\mathbb{C}}$ in $\mathbb{P}^n_{\mathbb{C}}$ as above we have

$$x(t) = (1 : a_1 + b_1 t : \cdots : a_n + b_n t) = \left(\frac{1}{t} : \frac{a_1}{t} + b_1 : \cdots : \frac{a_n}{t} + b_n\right)$$

in homogeneous coordinates, and thus (in a suitable topology) we get a limit point $(0 : b_1 : \cdots : b_n) \in \mathbb{P}^n_{\mathbb{C}} \backslash \mathbb{A}^n_{\mathbb{C}}$ at infinity for $x(t)$ as $t \to \infty$. This limit point obviously remembers the *direction*, but not the *position* of the original line. Hence we can say that in $\mathbb{P}^n$ we have added a point at infinity to $\mathbb{A}^n$ in each direction. In other words, after extension to $\mathbb{P}^n$ two distinct lines in $\mathbb{A}^n$ will meet at infinity if and only if they are parallel, i. e. point in the same direction.

Usually, it is more helpful to think of the projective space $\mathbb{P}^n$ as the affine space $\mathbb{A}^n$ compactified by adding some points (parametrized by $\mathbb{P}^{n-1}$) at infinity, rather than as the set of 1-dimensional linear subspaces in K^{n+1}. In fact, after having given $\mathbb{P}^n$ the structure of a variety we will see in Proposition 7.2 and Exercise 7.3 (b) that with the above constructions $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$ are open and closed subvarieties of $\mathbb{P}^n$, respectively.

Remark 6.4 ($\mathbb{P}^n_{\mathbb{C}}$ is compact in the classical topology)**.** In the case $K = \mathbb{C}$ one can give $\mathbb{P}^n_{\mathbb{C}}$ a standard (quotient) topology by declaring a subset $U \subset \mathbb{P}^n$ to be open if its inverse image under the quotient map $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ is open in the standard topology. Then $\mathbb{P}^n_{\mathbb{C}}$ is compact: let

$$S = \{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} : |x_0|^2 + \cdots + |x_n|^2 = 1\}$$

be the unit sphere in $\mathbb{C}^{n+1}$. This is a compact space as it is closed and bounded. Moreover, as every point in $\mathbb{P}^n$ can be represented by a unit vector in S, the restricted map $\pi|_S : S \to \mathbb{P}^n$ is surjective. Hence $\mathbb{P}^n$ is compact as a continuous image of a compact set.

Remark 6.5 (Homogeneous polynomials)**.** In complete analogy to affine varieties, we now want to define projective varieties to be subsets of $\mathbb{P}^n$ that can be given as the zero locus of some polynomials in the homogeneous coordinates. Note however that if $f \in K[x_0, \ldots, x_n]$ is an arbitrary polynomial, it does not make sense to write down a definition like

$$V(f) = \{(x_0 : \cdots : x_n) : f(x_0, \ldots, x_n) = 0\} \quad \subset \mathbb{P}^n,$$

because the homogeneous coordinates are only defined up to a common scalar. For example, if $f = x_1^2 - x_0 \in K[x_0, x_1]$ then $f(1,1) = 0$ and $f(-1,-1) \neq 0$, although $(1:1) = (-1:-1)$ in $\mathbb{P}^1$. To get rid of this problem we have to require that f is *homogeneous*, i. e. that all of its monomials have the same (total) degree d: in this case

$$f(\lambda x_0,\dots,\lambda x_n) = \lambda^d f(x_0,\dots,x_n) \text{ for all } \lambda \in K^*,$$

and so in particular we see that

$$f(\lambda x_0,\dots,\lambda x_n) = 0 \quad \Leftrightarrow \quad f(x_0,\dots,x_n) = 0,$$

so that the zero locus of f is well-defined in $\mathbb{P}^n$. So before we can start with our discussion of projective varieties we have to set up some algebraic language to be able to talk about homogeneous elements in a ring (or K-algebra).

Definition 6.6 (Graded rings and K-algebras)**.**

(a) A **graded ring** is a ring R together with Abelian subgroups $R_d \subset R$ for all $d \in \mathbb{N}$, such that:

- Every element $f \in R$ has a unique decomposition $f = \sum_{d\in\mathbb{N}} f_d$ such that $f_d \in R_d$ for all $d \in \mathbb{N}$ and only finitely many f_d are non-zero. In accordance with the direct sum notation in linear algebra, we usually write this condition as $R = \bigoplus_{d\in\mathbb{N}} R_d$.
- For all $d,e \in \mathbb{N}$ and $f \in R_d$, $g \in R_e$ we have $fg \in R_{d+e}$.

For $f \in R\backslash\{0\}$ the biggest number $d \in \mathbb{N}$ with $f_d \neq 0$ in the decomposition $f = \sum_{d\in\mathbb{N}} f_d$ as above is called the **degree** $\deg f$ of f. The elements of $R_d\backslash\{0\}$ are said to be **homogeneous** (of degree d). We call $f = \sum_{d\in\mathbb{N}} f_d$ and $R = \bigoplus_{d\in\mathbb{N}} R_d$ as above the **homogeneous decomposition** of f and R, respectively.

(b) If R is also a K-algebra in addition to (a), we say that it is a **graded K-algebra** if $\lambda f \in R_d$ for all $d \in \mathbb{N}$ and $f \in R_d$.

Example 6.7. The polynomial ring $R = K[x_0,\dots,x_n]$ is obviously a graded ring with

$$R_d = \left\{ \sum_{\substack{i_0,\dots,i_n\in\mathbb{N} \\ i_0+\cdots+i_n=d}} a_{i_0,\dots,i_n}\, x_0^{i_0}\cdot\,\cdots\,\cdot x_n^{i_n} : a_{i_0,\dots,i_n} \in K \text{ for all } i_0,\dots,i_n \right\}$$

for all $d \in \mathbb{N}$. In the following we will always consider it with this grading.

Exercise 6.8. Let $R \neq 0$ be a graded ring. Show that the multiplicative unit $1 \in R$ is homogeneous of degree 0.

Of course, we will also need ideals in graded rings. Naively, one might expect that we should consider ideals consisting of homogeneous elements in this case. However, as an ideal has to be closed under multiplication with *arbitrary* ring elements, it is virtually impossible that all of its elements are homogeneous. Instead, the correct notion of homogeneous ideal is the following.

Definition 6.9 (Homogeneous ideals)**.** An ideal in a graded ring is called **homogeneous** if it can be generated by homogeneous elements.

Lemma 6.10 (Properties of homogeneous ideals)**.** *Let I and J be ideals in a graded ring R.*

(a) *The ideal I is homogeneous if and only if for all $f \in I$ with homogeneous decomposition $f = \sum_{d\in\mathbb{N}} f_d$ we also have $f_d \in I$ for all d.*

(b) *If I and J are homogeneous then so are $I+J$, IJ, $I\cap J$, and $\sqrt{I}$.*

(c) *If I is homogeneous then the quotient R/I is a graded ring with homogeneous decomposition $R/I = \bigoplus_{d\in\mathbb{N}} R_d/(R_d\cap I)$.*

Proof.

(a) "$\Rightarrow$": Let $I = (h_j : j \in J)$ for homogeneous elements $h_j \in R$ for all j, and let $f \in I$. Then $f = \sum_{j\in J} g_j h_j$ for some (not necessarily homogeneous) $g_j \in R$, of which only finitely many are non-zero. If we denote by $g_j = \sum_{e \in \mathbb{N}} g_{j,e}$ the homogeneous decompositions of these elements, the degree-d part of f for $d \in \mathbb{N}$ is

$$f_d = \sum_{\substack{j \in J, e \in \mathbb{N} \\ e + \deg h_j = d}} g_{j,e} h_j \in I.$$

"$\Leftarrow$": Now let $I = (h_j : j \in J)$ for arbitrary elements $h_j \in R$ for all j. If $h_j = \sum_{d \in \mathbb{N}} h_{j,d}$ is their homogeneous decomposition, we have $h_{j,d} \in I$ for all j and d by assumption, and thus $I = (h_{j,d} : j \in J, d \in \mathbb{N})$ can be generated by homogeneous elements.

(b) If I and J are generated by homogeneous elements, then clearly so are $I + J$ (which is generated by $I \cup J$) and IJ. Moreover, I and J then satisfy the equivalent condition of (a), and thus so does $I \cap J$.

It remains to be shown that $\sqrt{I}$ is homogeneous. We will check the condition of (a) for any $f \in \sqrt{I}$ by induction over the degree d of f. Writing $f = f_0 + \cdots + f_d$ in its homogeneous decomposition, we get

$$f^n = (f_0 + \cdots + f_d)^n = f_d^n + (\text{terms of lower degree}) \quad \in I$$

for some $n \in \mathbb{N}$, hence $f_d^n \in I$ by (a), and thus $f_d \in \sqrt{I}$. But then $f - f_d = f_0 + \cdots + f_{d-1} \in \sqrt{I}$ as well, and so by the induction hypothesis we also see that $f_0, \ldots, f_{d-1} \in \sqrt{I}$.

(c) It is clear that $R_d/(R_d \cap I) \to R/I$, $\overline{f} \mapsto \overline{f}$ is an injective group homomorphism, so that we can consider $R_d/(R_d \cap I)$ as a subgroup of R/I for all d.

Now let $f \in R$ be arbitrary, with homogeneous decomposition $f = \sum_{d \in \mathbb{N}} f_d$. Then $\overline{f} = \sum_{d \in \mathbb{N}} \overline{f_d}$ with $\overline{f_d} \in R_d/(R_d \cap I)$, so $\overline{f}$ also has a homogeneous decomposition. Moreover, this decomposition is unique: if $\sum_{d \in \mathbb{N}} \overline{f_d} = \sum_{d \in \mathbb{N}} \overline{g_d}$ are two such decompositions of the same element in R/I then $\sum_{d \in \mathbb{N}} (f_d - g_d)$ lies in I, hence by (a) is of the form $\sum_{d \in \mathbb{N}} h_d$ with all $h_d \in R_d \cap I$. But then

$$\sum_{d \in \mathbb{N}} (f_d - g_d - h_d) = 0,$$

which implies that $f_d - g_d - h_d = 0$, and thus $\overline{f_d} = \overline{g_d} \in R_d/(R_d \cap I)$ for all d. □

With this preparation we can now define projective varieties in the same way as affine ones. For simplicity, for a homogeneous polynomial $f \in K[x_0, \ldots, x_n]$ and a point $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n$ we will write the condition $f(x_0, \ldots, x_n) = 0$ (which is well-defined by Remark 6.5) also as $f(x) = 0$.

Definition 6.11 (Projective varieties and their ideals)**.** Let $n \in \mathbb{N}$.

(a) Let $S \subset K[x_0, \ldots, x_n]$ be a set of homogeneous polynomials. Then the **(projective) zero locus** of S is defined as

$$V(S) := \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in S\} \quad \subset \mathbb{P}^n.$$

Subsets of $\mathbb{P}^n$ that are of this form are called **projective varieties**. For $S = (f_1, \ldots, f_k)$ we will write $V(S)$ also as $V(f_1, \ldots, f_k)$.

(b) For a homogeneous ideal $I \trianglelefteq K[x_0, \ldots, x_n]$ we set

$$V(I) := \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all homogeneous } f \in I\} \quad \subset \mathbb{P}^n.$$

Obviously, if I is the ideal generated by a set S of homogeneous polynomials then $V(I) = V(S)$.

(c) If $X \subset \mathbb{P}^n$ is any subset we define its **ideal** to be

$$I(X) := \big(f \in K[x_0, \ldots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } x \in X\big) \quad \trianglelefteq K[x_0, \ldots, x_n].$$

(Note that the homogeneous polynomials vanishing on X do not form an ideal yet, so that we have to take the ideal generated by them.)

If we want to distinguish these projective constructions from the affine ones in Definitions 1.2 (c) and 1.10 we will denote them by $V_p(S)$ and $I_p(X)$, and the affine ones by $V_a(S)$ and $I_a(X)$, respectively.

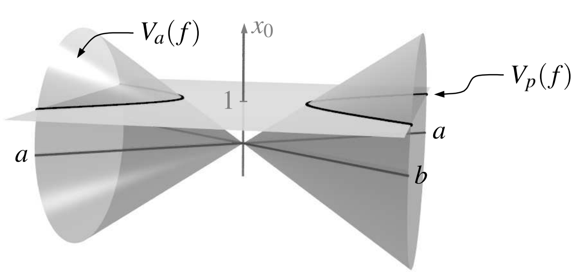
Example 6.12.

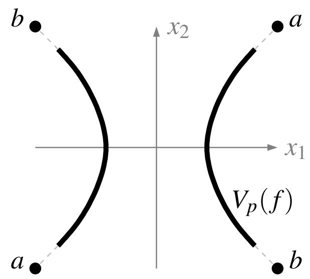
(a) As in the affine case, the empty set $\emptyset = V_p(1)$ and the whole space $\mathbb{P}^n = V_p(0)$ are projective varieties.

(b) If $f_1,\dots,f_r \in K[x_0,\dots,x_n]$ are homogeneous linear polynomials then $V_p(f_1,\dots,f_r) \subset \mathbb{P}^n$ is a projective variety. Projective varieties that are of this form are called **linear subspaces** of $\mathbb{P}^n$.

Exercise 6.13. Let $a \in \mathbb{P}^n$ be a point. Show that the one-point set $\{a\}$ is a projective variety, and compute explicit generators for the ideal $I_p(\{a\}) \trianglelefteq K[x_0,\dots,x_n]$.

Example 6.14. Let $f = x_1^2 - x_2^2 - x_0^2 \in \mathbb{C}[x_0,x_1,x_2]$. The real part of the affine zero locus $V_a(f) \subset \mathbb{A}^3$ of this homogeneous polynomial is the 2-dimensional cone shown in the picture below on the left. According to Definition 6.11, its projective zero locus $V_p(f) \subset \mathbb{P}^2$ is the set of all 1-dimensional linear subspaces contained in this cone — but we have seen in Remark 6.3 already that we should rather think of $\mathbb{P}^2$ as the affine plane $\mathbb{A}^2$ (embedded in $\mathbb{A}^3$ at $x_0 = 1$) together with some points at infinity. With this interpretation the real part of $V_p(f)$ consists of the hyperbola shown below on the right (whose equation $x_1^2 - x_2^2 - 1 = 0$ can be obtained by setting $x_0 = 1$ in f), together with two points a and b at infinity. In the 3-dimensional picture on the left, these two points correspond to the two 1-dimensional linear subspaces parallel to the plane at $x_0 = 1$, in the 2-dimensional picture of the affine part in $\mathbb{A}^2$ on the right they can be thought of as points at infinity in the corresponding directions. Note that, in the latter interpretation, "opposite" points at infinity are actually the same, since they correspond to the same 1-dimensional linear subspace in $\mathbb{C}^3$.





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We see in this example that the affine and projective zero locus of f carry essentially the same geometric information — the difference is just whether we consider the cone as a set of individual points, or as a union of 1-dimensional linear subspaces in $\mathbb{A}^3$. Let us now formalize and generalize this correspondence.

Definition 6.15 (Cones)**.** Let $\pi : \mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$, $(x_0,\dots,x_n) \mapsto (x_0 : \cdots : x_n)$.

(a) An affine variety $X \subset \mathbb{A}^{n+1}$ is called a **cone** if $0 \in X$, and $\lambda x \in X$ for all $\lambda \in K$ and $x \in X$. In other words, it consists of the origin together with a union of lines through 0.

(b) For a cone $X \subset \mathbb{A}^{n+1}$ we call

$$\mathbb{P}(X) := \pi(X\backslash\{0\}) = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n : (x_0,\dots,x_n) \in X\} \quad \subset \mathbb{P}^n$$

the **projectivization** of X.

(c) For a projective variety $X \subset \mathbb{P}^n$ we call

$$C(X) := \{0\} \cup \pi^{-1}(X) = \{0\} \cup \{(x_0,\dots,x_n) : (x_0 : \cdots : x_n) \in X\} \quad \subset \mathbb{A}^{n+1}$$

the **cone** over X (we will see in Remark 6.17 that this is in fact a cone in the sense of (a)).

Remark 6.16 (Cones and homogeneous ideals)**.** If $S \subset K[x_0,\dots,x_n]$ is a set of homogeneous polynomials with non-empty affine zero locus in $\mathbb{A}^{n+1}$ then $V_a(S)$ is a cone: clearly, we have $0 \in V_a(S)$ as every non-constant homogeneous polynomial vanishes at the origin. Moreover, let $\lambda \in K$ and $x \in V_a(S)$. Then $f(x) = 0$ for all $f \in S$, hence $f(\lambda x) = \lambda^{\deg f} f(x) = 0$, and so $\lambda x \in V_a(S)$ as well.

Conversely, the ideal $I(X)$ of a cone $X \subset \mathbb{A}^{n+1}$ is homogeneous: let $f \in I(X)$ with homogeneous decomposition $f = \sum_{d\in\mathbb{N}} f_d$. Then for all $x \in X$ we have $f(x) = 0$, and therefore also

$$0 = f(\lambda x) = \sum_{d\in\mathbb{N}} \lambda^d f_d(x)$$

for all $\lambda \in K$ since X is a cone. This means that we have the zero polynomial in λ, i. e. that $f_d(x) = 0$ for all d, and thus $f_d \in I(X)$. Hence $I(X)$ is homogeneous by Lemma 6.10 (a).

Remark 6.17 (Cones $\leftrightarrow$ projective varieties)**.** Let $S \subset K[x_0,\dots,x_n]$ be a set of homogeneous polynomials with non-empty affine zero locus in $\mathbb{A}^{n+1}$. Then $V_a(S)$ is a cone by Remark 6.16, and by construction we have

$$\mathbb{P}(V_a(S)) = V_p(S) \qquad \text{and} \qquad C(V_p(S)) = V_a(S).$$

In particular, the projectivization $\mathbb{P}(V_a(S))$ is a projective variety, and $C(V_p(S))$ is a cone. Moreover, as Remark 6.16 also shows that every cone is of the form $V_a(S)$ for a suitable set S of homogeneous polynomials (namely generators for its homogeneous ideal), we obtain a one-to-one correspondence

$$\begin{array}{rcl} \{\text{cones in } \mathbb{A}^{n+1}\} & \longleftrightarrow & \{\text{projective varieties in } \mathbb{P}^n\} \\ X & \longmapsto & \mathbb{P}(X) \\ C(X) & \longleftarrow\!\shortmid & X. \end{array}$$

In other words, the correspondence works by passing from the affine to the projective zero locus (and vice versa) of the same set of homogeneous polynomials, as in Example 6.14. Note that in this way linear subspaces of $\mathbb{A}^{n+1}$ correspond exactly to linear subspaces of $\mathbb{P}^n$ in the sense of Example 6.12 (b).

Having defined projective varieties, we can now proceed with their study as in the affine case. First of all, we should associate a coordinate ring to a projective variety, and consider zero loci and ideals with respect to these coordinate rings.

Construction 6.18 (Relative version of zero loci and ideals)**.** Let $Y \subset \mathbb{P}^n$ be a projective variety. In analogy to Definition 1.19 we call

$$S(Y) := K[x_0,\dots,x_n]/I(Y)$$

the **homogeneous coordinate ring** of Y. By Lemma 6.10 (c) it is a graded ring, so that it makes sense to talk about homogeneous elements of $S(Y)$. Moreover, the condition $f(x) = 0$ is still well-defined for a homogeneous element $f \in S(Y)$ and a point $x \in Y$, and thus we can define as in Definition 6.11

$$V(I) := \{x \in Y : f(x) = 0 \text{ for all homogeneous } f \in I\} \qquad \text{for a homogeneous ideal } I \trianglelefteq S(Y)$$

(and similarly for a set of homogeneous polynomials in $S(Y)$), and

$$I(X) := \big(f \in S(Y) \text{ homogeneous} : f(x) = 0 \text{ for all } x \in X\big) \qquad \text{for a subset } X \subset Y.$$

As before, in case of possible confusion we will decorate V and I with the subscript Y and/or p to denote the relative and projective situation, respectively. Subsets of Y that are of the form $V_Y(I)$ for a homogeneous ideal $I \trianglelefteq S(Y)$ will be called **projective subvarieties** of Y; these are obviously exactly the projective varieties contained in Y.

Remark 6.19. Let Y be a projective variety. The following results are completely analogous to the affine case:

(a) (*Hilbert's Basis Theorem*) Every homogeneous ideal in $S(Y)$ can be generated by finitely many elements. In fact, it is finitely generated by [G5, Proposition 7.13 and Remark 7.15], and hence also by homogeneous elements as we have seen in the proof of part "$\Leftarrow$" of Lemma 6.10 (a).

(b) The operations $V_Y(\cdot)$ and $I_Y(\cdot)$ reverse inclusions, we have $X = V_Y(I_Y(X))$ for every projective subvariety X of Y, and $J \subset I_Y(V_Y(J))$ for any homogeneous ideal $J \trianglelefteq S(Y)$ — these statements follow literally in the same way as in Lemma 1.12.

(c) The ideal $I_Y(X)$ of a projective subvariety $X \subset Y$ is radical: by Lemma 6.10 (b) the radical $\sqrt{I_Y(X)}$ is homogeneous, so it suffices to prove that $f \in \sqrt{I_Y(X)}$ implies $f \in I_Y(X)$ for any homogeneous f. But this is obvious since $f^k = 0$ on Y for some k implies $f = 0$ on Y.

(d) By (c) the ideal $I_p(Y) \trianglelefteq K[x_0,\dots,x_n]$ is radical. Hence Proposition 1.17 implies that it is also the ideal of its affine zero locus $V_a(I_p(Y)) \subset \mathbb{A}^{n+1}$. But $V_p(I_p(Y)) = Y$ by (b), and so we see by Remark 6.17 that $V_a(I_p(Y)) = C(Y)$. Therefore we conclude that $I_p(Y) = I_a(C(Y))$, and thus that $S(Y) = A(C(Y))$. Hence every homogeneous coordinate ring of a projective variety can also be interpreted as a usual coordinate ring of an affine variety.

Remark 6.20. A remark that is sometimes useful is that every projective subvariety X of a projective variety $Y \subset \mathbb{P}^n$ can be written as the zero locus of finitely many homogeneous polynomials in $S(Y)$ *of the same degree*. This follows easily from the fact that $V_p(f) = V_p(x_0^d f,\dots,x_n^d f)$ for all homogeneous $f \in S(Y)$ and every $d \in \mathbb{N}$. However, it is not true that every homogeneous ideal in $S(Y)$ can be generated by homogeneous elements of the same degree.

Of course, we would also expect a projective version of the Nullstellensatz as in Proposition 1.21 (b), i. e. that $I_Y(V_Y(J)) = \sqrt{J}$ for any homogeneous ideal J in the homogeneous coordinate ring of a projective variety Y. This is *almost* true and can in fact be proved by reduction to the affine case — there is one exception however, since the origin in $\mathbb{A}^{n+1}$ does not correspond to a point in projective space $\mathbb{P}^n$:

Example 6.21 (Irrelevant ideal)**.** Let $Y \subset \mathbb{P}^n$ be a non-empty projective variety, and let

$$I_0 := (\overline{x_0},\dots,\overline{x_n}) \trianglelefteq S(Y) = K[x_0,\dots,x_n]/I(Y).$$

Then I_0 is a homogeneous radical ideal, and its projective zero locus is empty since there is no point in Y all of whose homogeneous coordinates are zero. Hence

$$I_Y(V_Y(I_0)) = I_Y(\emptyset) = S(Y),$$

which is not equal to $\sqrt{I_0} = I_0$. In fact, I_0 can never appear as the ideal of a projective variety, since $I_0 = I_Y(X)$ for some $X \subset Y$ would imply $X = V_Y(I_Y(X)) = V_Y(I_0) = \emptyset$ by Remark 6.19 (b), in contradiction to $I_Y(\emptyset) = S(Y)$.

We will see now however that this is the only counterexample to the projective version of the Nullstellensatz. The ideal I_0 above is therefore often called the **irrelevant ideal**. Note that by Proposition 1.21 (b) this is the unique radical ideal whose affine zero locus is $\{0\}$.

Proposition 6.22 (Projective Nullstellensatz). *Let Y be a non-empty projective variety, and let $J \subset S(Y)$ be a homogeneous ideal such that $\sqrt{J}$ is not the irrelevant ideal. Then $I_p(V_p(J)) = \sqrt{J}$. In particular, we have an inclusion-reversing one-to-one correspondence*

$$\begin{array}{rcl} \{\textit{projective subvarieties of } Y\} & \longleftrightarrow & \left\{ \begin{array}{c} \textit{homogeneous radical ideals in } S(Y) \\ \textit{not equal to the irrelevant ideal} \end{array} \right\} \\ X & \longmapsto & I_p(X) \\ V_p(J) & \longleftarrow\!\shortmid & J. \end{array}$$

Proof. In this proof we will regard J as an ideal in both $S(Y)$ and $A(C(Y))$ (see Remark 6.19 (d)), so that we can take both its projective and its affine zero locus. Note then that

$$\begin{aligned} I_p(V_p(J)) &= \big(f \in S(Y) \text{ homogeneous} : f(x) = 0 \text{ for all } x \in V_p(J)\big) \\ &= \big(f \in S(Y) \text{ homogeneous} : f(x) = 0 \text{ for all } x \in V_a(J)\backslash\{0\}\big). \end{aligned}$$

As the affine zero locus of polynomials is closed, we can rewrite this as

$$I_p(V_p(J)) = \big(f \in S(Y) \text{ homogeneous} : f(x) = 0 \text{ for all } x \in \overline{V_a(J)\backslash\{0\}}\big).$$

By Example 6.21 we have $V_a(J) = V_a(\sqrt{J}) \neq \{0\}$ since $\sqrt{J}$ is not the irrelevant ideal. But then $\overline{V_a(J)\backslash\{0\}} = V_a(J)$: if $V_a(J) = \emptyset$ this is trivial, and otherwise the cone $V_a(J)\backslash\{0\}$ contains at least one line without the origin, so that the origin lies in its closure. Hence we get

$$I_p(V_p(J)) = \big(f \in S(Y) \text{ homogeneous} : f(x) = 0 \text{ for all } x \in V_a(J)\big).$$

As the ideal of the cone $V_a(J)$ is homogeneous by Remark 6.16 this can be rewritten as $I_p(V_p(J)) = I_a(V_a(J))$, which is equal to $\sqrt{J}$ by the affine Nullstellensatz of Proposition 1.21 (b).

The one-to-one correspondence then follows together with the statement $V_p(I_p(X)) = X$ from Remark 6.19 (b) (note that $I_p(X)$ is always homogeneous by definition, radical by Remark 6.19 (c), and not equal to the irrelevant ideal by Example 6.21). □

Remark 6.23 (Properties of $V_p(\cdot)$ and $I_p(\cdot)$)**.** The operations $V_p(\cdot)$ and $I_p(\cdot)$ satisfy the same properties as their affine counterparts in Lemma 1.24, Remark 1.25, and Lemma 1.26. More precisely, for any projective variety X we have:

(a) For any family $\{S_i\}$ of subsets of $S(X)$ we have $\bigcap_i V_p(S_i) = V_p(\bigcup_i S_i)$; for any two subsets $S_1, S_2 \subset S(X)$ we have $V_p(S_1) \cup V_p(S_2) = V_p(S_1 S_2)$.

(b) If $J_1, J_2 \trianglelefteq S(X)$ are homogeneous ideals then

$$V_p(J_1) \cap V_p(J_2) = V_p(J_1 + J_2) \quad \text{and} \quad V_p(J_1) \cup V_p(J_2) = V_p(J_1 J_2) = V_p(J_1 \cap J_2).$$

(c) For subsets Y_1, Y_2 of a projective variety X we have $I_p(Y_1 \cup Y_2) = I_p(Y_1) \cap I_p(Y_2)$. Moreover, $I_p(Y_1 \cap Y_2) = \sqrt{I_p(Y_1) + I_p(Y_2)}$ unless the latter is the irrelevant ideal (which is only possible if Y_1 and Y_2 are disjoint).

The proof of these statements is completely analogous to the affine case.

In particular, by (a) it follows that arbitrary intersections and finite unions of projective subvarieties of X are again projective subvarieties, and hence we can define the Zariski topology on X in the same way as in the affine case:

Definition 6.24 (Zariski topology)**.** The **Zariski topology** on a projective variety X is the topology whose closed sets are exactly the projective subvarieties of X, i. e. the subsets of the form $V_p(S)$ for some set $S \subset S(X)$ of homogeneous elements.

Of course, from now on we will always use this topology for projective varieties and their subsets. Note that, in the same way as in Remark 2.3, this is well-defined in the sense that the Zariski topology on a projective variety $X \subset \mathbb{P}^n$ agrees with the subspace topology of X in $\mathbb{P}^n$. Moreover, since we want to consider $\mathbb{A}^n$ as a subset of $\mathbb{P}^n$ as in Remark 6.3 we should also check that the Zariski topology on $\mathbb{A}^n$ is the same as the subspace topology of $\mathbb{A}^n$ in $\mathbb{P}^n$. To do this, we need the following definition.

Definition 6.25 (Homogenization)**.**

(a) Let

$$f = \sum_{i_1,\dots,i_n \in \mathbb{N}} a_{i_1,\dots,i_n}\, x_1^{i_1} \cdot \cdots \cdot x_n^{i_n} \quad \in K[x_1,\dots,x_n]$$

be a (non-zero) polynomial of degree d. We define its **homogenization** to be

$$\begin{aligned} f^h &:= x_0^d\, f\Big(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\Big) \\ &= \sum_{i_1,\dots,i_n \in \mathbb{N}} a_{i_1,\dots,i_n}\, x_0^{d-i_1-\cdots-i_n} x_1^{i_1} \cdot \cdots \cdot x_n^{i_n} \quad \subset K[x_0,\dots,x_n]; \end{aligned}$$

obviously this is a homogeneous polynomial of degree d.

(b) The **homogenization** of an ideal $I \trianglelefteq K[x_1,\dots,x_n]$ is defined to be the ideal I^h in $K[x_0,\dots,x_n]$ generated by all f^h for $f \in I$.

Example 6.26. For $f = x_1^2 - x_2^2 - 1 \in K[x_1,x_2]$ we have $f^h = x_1^2 - x_2^2 - x_0^2 \in K[x_0,x_1,x_2]$.

Remark 6.27. If $f,g \in K[x_1,\dots,x_n]$ are polynomials of degree d and e, respectively, then fg has degree $d+e$, and so we get

$$(fg)^h = x_0^{d+e} f\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right) \cdot g\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right) = f^h \cdot g^h.$$

However, $(f+g)^h$ is clearly not equal to $f^h + g^h$ in general — in fact, $f^h + g^h$ is usually not even homogeneous. This is the reason why in Definition 6.25 (b) we have to take the ideal generated by all homogenizations of polynomials in I, instead of just all these homogenizations themselves.

Remark 6.28 ($\mathbb{A}^n$ as an open subset of $\mathbb{P}^n$)**.** Recall from Remark 6.3 that we want to identify the subset $U_0 = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n : x_0 \neq 0\}$ of $\mathbb{P}^n$ with $\mathbb{A}^n$ by the bijective map

$$F : \mathbb{A}^n \to U_0,\ (x_1,\dots,x_n) \mapsto (1 : x_1 : \cdots : x_n).$$

Obviously, U_0 is an open subset of $\mathbb{P}^n$. Moreover, with the above identification the subspace topology of $U_0 = \mathbb{A}^n \subset \mathbb{P}^n$ is the affine Zariski topology:

(a) If $X = V_p(S) \cap \mathbb{A}^n$ is closed in the subspace topology (for a subset $S \subset K[x_0,\dots,x_n]$ of homogeneous polynomials) then $X = V(f(1,\cdot) : f \in S)$ is also Zariski closed.

(b) If $X = V(S) \subset \mathbb{A}^n$ is Zariski closed (with $S \subset K[x_1,\dots,x_n]$) then $X = V_p(f^h : f \in S) \cap \mathbb{A}^n$ is closed in the subspace topology as well.

In other words we can say that the map $F : \mathbb{A}^n \to U_0$ above is a homeomorphism. In fact, after having given $\mathbb{P}^n$ the structure of a variety we will see in Proposition 7.2 that it is even an isomorphism of varieties.

Having defined the Zariski topology on projective varieties (or more generally on subsets of $\mathbb{P}^n$) we can now immediately apply all topological concepts of Chapter 2 to this new situation. In particular, the notions of connectedness, irreducibility, and dimension are well-defined for projective varieties (and have the same geometric interpretation as in the affine case). Let us study some examples using these concepts.

Remark 6.29 ($\mathbb{P}^n$ is irreducible of dimension n)**.** Of course, by symmetry of the coordinates, it follows from Remark 6.28 that all subsets $U_i = \{(x_0 : \cdots : x_n) : x_i \neq 0\}$ of $\mathbb{P}^n$ for $i = 0,\dots,n$ are homeomorphic to $\mathbb{A}^n$ as well. As these subsets cover $\mathbb{P}^n$ and have non-empty intersections, we conclude by Exercise 2.20 (b) that $\mathbb{P}^n$ is irreducible, and by Exercise 2.33 (a) that $\dim \mathbb{P}^n = n$.

Exercise 6.30. Let $L_1, L_2 \subset \mathbb{P}^3$ be two disjoint lines (i. e. 1-dimensional linear subspaces in the sense of Example 6.12 (b)), and let $a \in \mathbb{P}^3 \backslash (L_1 \cup L_2)$. Show that there is a unique line $L \subset \mathbb{P}^3$ through a that intersects both L_1 and L_2.

Is the corresponding statement for lines and points in $\mathbb{A}^3$ true as well?

Exercise 6.31.

(a) Prove that a graded ring R is an integral domain if and only if for all *homogeneous* elements $f,g \in R$ with $fg = 0$ we have $f = 0$ or $g = 0$.

(b) Show that a projective variety X is irreducible if and only if its homogeneous coordinate ring $S(X)$ is an integral domain.

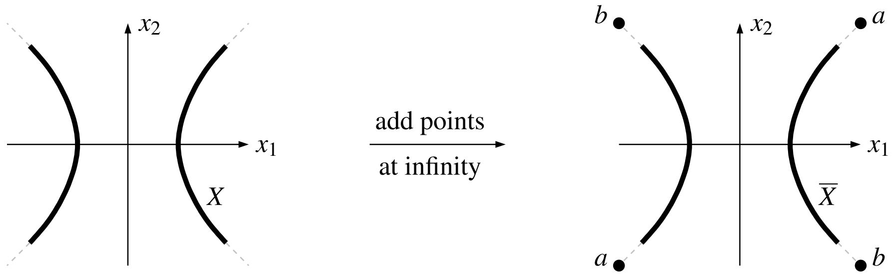
Exercise 6.32. In this exercise we want to show that an intersection of projective varieties is never empty unless one would expect it to be empty for dimensional reasons — so e. g. the phenomenon of parallel non-intersecting lines in the plane does not occur in projective space (which we have seen already in Remark 6.3).

So let $X,Y \subset \mathbb{P}^n$ be non-empty projective varieties. Show:

(a) The dimension of the cone $C(X) \subset \mathbb{A}^{n+1}$ is $\dim X + 1$.

(b) If $\dim X + \dim Y \geq n$ then $X \cap Y \neq \emptyset$.

We have just seen in Remark 6.28 (b) that we can use homogenizations of polynomials to describe an affine variety $X \subset \mathbb{A}^n$ in terms of their homogeneous coordinates on $\mathbb{A}^n \subset \mathbb{P}^n$. Let us now finish this chapter by showing that this construction can also be used to compute the closure of X in $\mathbb{P}^n$. As this will be a "compact" space in the sense of Remarks 6.3 and 6.4 we can think of this closure $\overline{X}$ as being obtained by compactifying X by some "points at infinity". For example, if we start with the affine hyperbola $X = V_a(x_1^2 - x_2^2 - 1) \subset \mathbb{A}^2$ in the picture below on the left, its closure $\overline{X} \subset \mathbb{P}^2$ adds the two points a and b at infinity as in Example 6.14.



We know already in this example that $\overline{X} = V_p(x_1^2 - x_2^2 - x_0^2)$, i. e. that the closure is the projective zero locus of the homogenization of the original polynomial $x_1^2 - x_2^2 - 1$. Let us now prove the corresponding general statement.

Proposition 6.33 (Computation of the projective closure)**.** *Let $I \trianglelefteq K[x_1, \dots, x_n]$ be an ideal. Consider its affine zero locus $X = V_a(I) \subset \mathbb{A}^n$, and its closure $\overline{X}$ in $\mathbb{P}^n$.*

(a) *We have $\overline{X} = V_p(I^h)$.*

(b) *If $I = (f)$ is a principal ideal then $\overline{X} = V_p(f^h)$.*

Proof.

(a) Clearly, the set $V_p(I^h)$ is closed and contains X: if $x = (x_1, \dots, x_n) \in X$ then $f(x_1, \dots, x_n) = 0$ and thus $f^h(1, x_1, \dots, x_n) = 0$ for all $f \in I$, which implies that $(1 : x_1 : \dots : x_n) \in V_p(I^h)$.

In order to show that $V_p(I^h)$ is the smallest closed set containing X let $Y \supset X$ be any closed set; we have to prove that $Y \supset V_p(I^h)$. As Y is closed we have $Y = V_p(J)$ for some homogeneous ideal J. Now any homogeneous element of J can be written as $x_0^d f^h$ for some $d \in \mathbb{N}$ and $f \in K[x_1, \dots, x_n]$, and for this element we have

$$
\begin{array}{rll}
& x_0^d f^h \text{ is zero on } X & (X \text{ is a subset of } Y) \\
\Rightarrow & f \text{ is zero on } X & (x_0 \neq 0 \text{ on } X \subset \mathbb{A}^n) \\
\Rightarrow & f \in I_a(X) = I_a(V_a(I)) = \sqrt{I} & (\text{Proposition 1.17}) \\
\Rightarrow & f^m \in I \text{ for some } m \in \mathbb{N} & \\
\Rightarrow & (f^h)^m = (f^m)^h \in I^h \text{ for some } m \in \mathbb{N} & (\text{Remark 6.27}) \\
\Rightarrow & f^h \in \sqrt{I^h} & \\
\Rightarrow & x_0^d f^h \in \sqrt{I^h}. &
\end{array}
$$

We therefore conclude that $J \subset \sqrt{I^h}$, and so $Y = V_p(J) \supset V_p(\sqrt{I^h}) = V_p(I^h)$ as desired.

(b) As $(f) = \{fg : g \in K[x_1, \dots, x_n]\}$, we have

$$\overline{X} = V((fg)^h : g \in K[x_1, \dots, x_n]) = V(f^h g^h : g \in K[x_1, \dots, x_n]) = V(f^h)$$

by (a) and Remark 6.27. □ 10

Example 6.34. In contrast to Proposition 6.33 (b), for general ideals it usually does not suffice to only homogenize a set of generators. As an example, consider the ideal $I = (x_1, x_2 - x_1^2) \trianglelefteq K[x_1, x_2]$ with affine zero locus $X = V_a(I) = \{0\} \subset \mathbb{A}^2$. This one-point set is also closed in $\mathbb{P}^2$, and thus

$\overline{X} = \{(1:0:0)\}$ is just the corresponding point in homogeneous coordinates. But if we homogenize the two given generators of I we obtain the homogeneous ideal $(x_1, x_0x_2 - x_1^2)$ with projective zero locus $\{(1:0:0),(0:0:1)\} \supsetneq \overline{X}$.

For those of you who know some computer algebra: one can show however that it suffices to homogenize a *Gröbner basis* of I. This makes the problem of finding $\overline{X}$ computationally feasible since in contrast to Proposition 6.33 (a) we only have to homogenize finitely many polynomials.

Exercise 6.35. Sketch the set of real points of the complex affine curve $X = V(x_1^3 - x_1x_2^2 + 1) \subset \mathbb{A}^2_{\mathbb{C}}$ and compute the points at infinity of its projective closure $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$.