5. Varieties

In this chapter we will now finally introduce the main objects of study in this class, the so-called *varieties*. As already announced in Example 4.18 they will be spaces that are not necessarily affine varieties themselves, but that can be covered by affine varieties. This idea is completely analogous to the definition of manifolds: recall that to construct them one first considers open subsets of $\mathbb{R}^n$ that are supposed to form the patches of your space, and then defines a manifold to be a topological space that looks locally like these patches. In our algebraic case we can say that the affine varieties form the basic patches of the spaces that we want to consider, and that general varieties are then spaces that look locally like affine varieties.

One of the main motivations for this generalization is that in the classical topology affine varieties over $\mathbb{C}$ are never bounded, and hence never compact, unless they are a finite set (see Exercise 2.34 (c)). As compact spaces are usually better-behaved than non-compact ones, it is therefore desirable to have a method to compactify an affine variety by "adding some points at infinity". Technically, this can be achieved by gluing it to other affine varieties that contain the points at infinity. The complete space can then obviously be covered by affine varieties. We will see this explicitly in Examples 5.5 (a) and 5.6, and much more generally when we construct projective varieties in Chapters 6 and 7.

So let us start by defining spaces that can be covered by affine varieties. They are called *prevarieties* since we will want to impose another condition on them later in Definition 5.19, which will then make them into varieties.

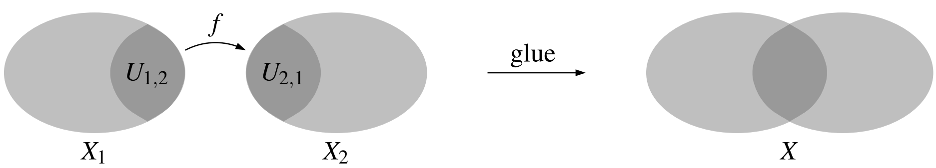
Definition 5.1 (Prevarieties)**.** A **prevariety** is a ringed space X that has a finite open cover by affine varieties. Morphisms of prevarieties are simply morphisms as ringed spaces. In accordance with Definition 3.1, the elements of $\mathscr{O}_X(U)$ for an open subset $U \subset X$ will be called **regular functions** on U.

Remark 5.2. Note that the open cover in Definition 5.1 is not part of the data needed to specify a prevariety — it is just required that such a cover exists. Any open subset of a prevariety that is an affine variety is called an *affine open set*.

Example 5.3. Of course, any affine variety is a prevariety. More generally, every open subset of an affine variety is a prevariety: it has a finite open cover by distinguished open subsets by Remark 3.9 (b), and these are affine open sets by Proposition 4.17.

The basic way to construct new prevarieties is to glue them together from previously known patches. For simplicity, let us start with the case when we only have two spaces to glue.

Construction 5.4 (Gluing two prevarieties)**.** Let X_1, X_2 be two prevarieties (e. g. affine varieties), and let $U_{1,2} \subset X_1$ and $U_{2,1} \subset X_2$ be non-empty open subsets. Moreover, let $f : U_{1,2} \to U_{2,1}$ be an isomorphism. Then we can define a prevariety X obtained by *gluing* X_1 and X_2 along f, as shown in the picture below:



- As a set, the space X is just the disjoint union $X_1 \cup X_2$ modulo the equivalence relation given by $a \sim f(a)$ and $f(a) \sim a$ for all $a \in U_{1,2}$ (in addition to $a \sim a$ for all $a \in X_1 \cup X_2$). Note that there are then natural embeddings $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$ that map a point to its equivalence class in $X_1 \cup X_2$.

- As a topological space, we call a subset $U \subset X$ open if $i_1^{-1}(U) \subset X_1$ and $i_2^{-1}(U) \subset X_2$ are open. In topology, this is usually called the *quotient topology* of the two maps i_1 and i_2.
- As a ringed space, we define the structure sheaf $\mathscr{O}_X$ by

$$\mathscr{O}_X(U) = \{\varphi : U \to K : \ i_1^* \varphi \in \mathscr{O}_{X_1}(i_1^{-1}(U)) \text{ and } i_2^* \varphi \in \mathscr{O}_{X_2}(i_2^{-1}(U))\}$$

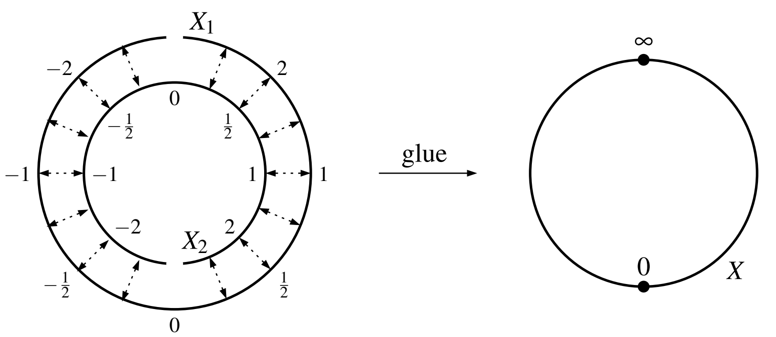
  for all open subsets $U \subset X$, where i_1^* and i_2^* denote the pull-backs of Definition 4.3 (a). Intuitively, this means that a function on the glued space is regular if it is regular when restricted to both patches. It is obvious that this defines a sheaf on X.

With this construction it is checked immediately that the images of i_1 and i_2 are open subsets of X that are isomorphic to X_1 and X_2. We will often drop the inclusion maps from the notation and say that X_1 and X_2 are open subsets of X. Since X_1 and X_2 can be covered by affine open subsets, the same is true for X, and thus X is a prevariety.

Example 5.5. As the simplest example of the above gluing construction, let $X_1 = X_2 = \mathbb{A}^1$ and $U_{1,2} = U_{2,1} = \mathbb{A}^1 \backslash \{0\}$ in the notation of Construction 5.4. We consider two different choices of gluing isomorphism $f : U_{1,2} \to U_{2,1}$:

(a) Let $f : U_{1,2} \to U_{2,1}$, $x \mapsto \frac{1}{x}$. By construction, the affine line $X_1 = \mathbb{A}^1$ is an open subset of X. Its complement $X \backslash X_1 = X_2 \backslash U_{2,1}$ is a single point that corresponds to 0 in X_2 and therefore to "$\infty = \frac{1}{0}$" in the coordinate of X_1. Hence we can think of the glued space X as $\mathbb{A}^1 \cup \{\infty\}$, and thus as a "compactification" of the affine line. We denote it by $\mathbb{P}^1$; it is a special case of the projective spaces that we will introduce in Chapter 6 (see Exercise 7.3 (a)).

In the case $K = \mathbb{C}$ the resulting space X is just the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Its subset of real points is shown in the picture below (with the dotted arrows indicating the gluing isomorphism), it is homeomorphic to a circle.

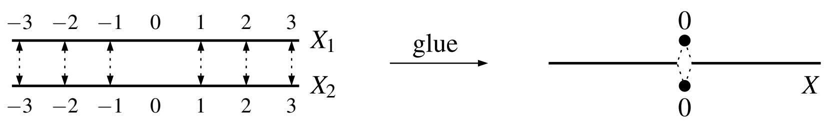


As an example of gluing morphisms as in Lemma 4.6, the morphisms

$$X_1 \to X_2 \subset \mathbb{P}^1, \ x \mapsto x \quad \text{and} \quad X_2 \to X_1 \subset \mathbb{P}^1, \ x \mapsto x$$

(that correspond to a reflection across the horizontal axis in the picture above) glue together to a single morphism $\mathbb{P}^1 \to \mathbb{P}^1$ that can be thought of as $x \mapsto \frac{1}{x}$ in the interpretation of $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$.

(b) Let $f : U_{1,2} \to U_{2,1}$ be the identity map. Then the space X obtained by gluing X_1 and X_2 along f is shown in the picture below, it is "an affine line with two zero points".



Obviously this is a somewhat weird space. Speaking in analytic terms in the case $K = \mathbb{C}$, a sequence of points tending to zero would have two possible limits in X, namely the two zero points. Also, as in (a) the two morphisms

$$X_1 \to X_2 \subset X, \ x \mapsto x \quad \text{and} \quad X_2 \to X_1 \subset X, \ x \mapsto x$$

glue again to a morphism $g : X \to X$; this time it exchanges the two zero points and thus leads to the set $\{x \in X : g(x) = x\} = \mathbb{A}^1 \backslash \{0\}$ not being closed in X, although it is given by an equality of continuous maps.

Usually we want to exclude such spaces from the objects we consider. In Definition 5.19 we will therefore impose an additional condition on our prevarieties that ensures that the above phenomena do not occur (see e. g. Proposition 5.21 (b)).

Example 5.6. Consider again the complex affine curve

$$X = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{C}} : x_2^2 = (x_1 - 1)(x_1 - 2) \cdots (x_1 - 2n)\}$$

of Example 0.1. We have already seen in the introduction that X can (and should) be "compactified" by adding two points at infinity, corresponding to the limit $x_1 \to \infty$ and the two possible values for x_2. Let us now construct this compactified space rigorously as a prevariety.

To be able to add a limit point "$x_1 = \infty$" to our space, let us make a coordinate change $\tilde{x}_1 = \frac{1}{x_1}$ (where $x_1 \neq 0$), so that the equation of the curve becomes

$$x_2^2 \tilde{x}_1^{2n} = (1 - \tilde{x}_1)(1 - 2\tilde{x}_1) \cdots (1 - 2n\tilde{x}_1).$$

If we make an additional coordinate change $\tilde{x}_2 = x_2 \tilde{x}_1^n$, this becomes

$$\tilde{x}_2^2 = (1 - \tilde{x}_1)(1 - 2\tilde{x}_1) \cdots (1 - 2n\tilde{x}_1).$$

In these coordinates we can now add our two points at infinity, as they correspond to $\tilde{x}_1 = 0$ (and therefore $\tilde{x}_2 = \pm 1$).

Hence the "compactified curve" of Example 0.1 can be constructed as the prevariety obtained by gluing the two affine varieties

$$\begin{aligned} & X_1 = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{C}} : x_2^2 = (x_1 - 1)(x_1 - 2) \cdots (x_1 - 2n)\} = X \\ \text{and} \quad & X_2 = \{(\tilde{x}_1, \tilde{x}_2) \in \mathbb{A}^2_{\mathbb{C}} : \tilde{x}_2^2 = (1 - \tilde{x}_1)(1 - 2\tilde{x}_1) \cdots (1 - 2n\tilde{x}_1)\} \end{aligned}$$

along the isomorphism

$$f : U_{1,2} \to U_{2,1}, \ \ (x_1, x_2) \mapsto (\tilde{x}_1, \tilde{x}_2) = \left(\frac{1}{x_1}, \frac{x_2}{x_1^n} \right)$$

with inverse

$$f^{-1} : U_{2,1} \to U_{1,2}, \ \ (\tilde{x}_1, \tilde{x}_2) \mapsto (x_1, x_2) = \left(\frac{1}{\tilde{x}_1}, \frac{\tilde{x}_2}{\tilde{x}_1^n} \right),$$

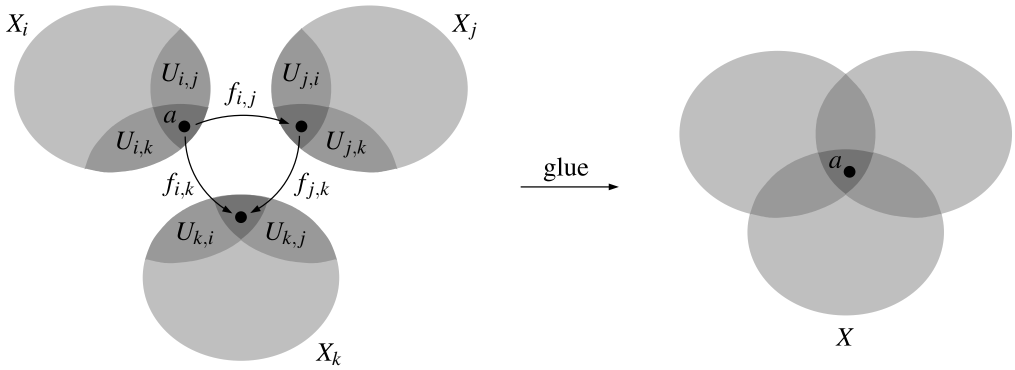
where $U_{1,2} = \{(x_1, x_2) : x_1 \neq 0\} \subset X_1$ and $U_{2,1} = \{(\tilde{x}_1, \tilde{x}_2) : \tilde{x}_1 \neq 0\} \subset X_2$.

Let us now turn to the general construction to glue more than two spaces together. In principle this works in the same way as in Construction 5.4; we just have an additional technical compatibility condition on the gluing isomorphisms.

Construction 5.7 (General gluing construction)**.** For a finite index set I let X_i be a prevariety for all $i \in I$. Moreover, as in the picture below suppose that for all $i, j \in I$ with $i \neq j$ we are given open subsets $U_{i,j} \subset X_i$ and isomorphisms $f_{i,j} : U_{i,j} \to U_{j,i}$ such that for all distinct $i, j, k \in I$ we have

(a) $f_{j,i} = f_{i,j}^{-1}$;

(b) $U_{i,j} \cap f_{i,j}^{-1}(U_{j,k}) \subset U_{i,k}$, and $f_{j,k} \circ f_{i,j} = f_{i,k}$ on $U_{i,j} \cap f_{i,j}^{-1}(U_{j,k})$.



In analogy to Construction 5.4 we can then define a set X by taking the disjoint union of all X_i for $i \in I$, modulo the equivalence relation $a \sim f_{i,j}(a)$ for all $a \in U_{i,j} \subset X_i$ (in addition to $a \sim a$ for all a). In fact, the conditions (a) and (b) above ensure precisely that this relation is symmetric and transitive, respectively. It is obvious that we can now make X into a prevariety by defining its topology and structure sheaf in the same way as in Construction 5.4. We call it the prevariety obtained by *gluing* the X_i along the isomorphisms $f_{i,j}$.

Exercise 5.8. Show:

(a) Every morphism $f : \mathbb{A}^1 \backslash \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^1 \to \mathbb{P}^1$.

(b) Not every morphism $f : \mathbb{A}^2 \backslash \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^2 \to \mathbb{P}^1$.

(c) Every morphism $f : \mathbb{P}^1 \to \mathbb{A}^1$ is constant.

Exercise 5.9.

(a) Show that every isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form $f(x) = \frac{ax+b}{cx+d}$ for some $a,b,c,d \in K$, where x is an affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$.

(b) Given three distinct points $a_1,a_2,a_3 \in \mathbb{P}^1$ and three distinct points $b_1,b_2,b_3 \in \mathbb{P}^1$, show that there is a unique isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(a_i) = b_i$ for $i = 1,2,3$.

Exercise 5.10. If X and Y are affine varieties we have seen in Proposition 3.10 and Corollary 4.8 that there is a one-to-one correspondence

$$\{\text{morphisms } X \to Y\} \longleftrightarrow \{K\text{-algebra homomorphisms } \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)\}$$
$$f \longmapsto f^*.$$

Does this statement still hold

(a) if X is an arbitrary prevariety (but Y is still affine);

(b) if Y is an arbitrary prevariety (but X is still affine)?

07

We have just seen how we can construct prevarieties by gluing affine varieties. For the rest of the chapter let us now study some of their basic properties. Of course, all topological concepts (like connectedness, irreducibility, and dimension) carry over immediately to the case of prevarieties. The irreducible decomposition of Proposition 2.15 is also applicable since a prevariety is always Noetherian:

Exercise 5.11. Prove:

(a) Any prevariety is a Noetherian topological space.

(b) If $X = X_1 \cup \cdots \cup X_m$ is the irreducible decomposition of a prevariety X, then the local dimension $\operatorname{codim}_X\{a\}$ of X at any point $a \in X$ is

$$\operatorname{codim}_X\{a\} = \max\{\dim X_i : a \in X_i\}.$$

(c) The statement of (a) would be false if we had defined a prevariety to be a ringed space that has an arbitrary (not necessarily finite) open cover by affine varieties.

As for properties of prevarieties involving the structure as ringed spaces, we should first of all figure out to what extent their subsets, images and inverse images under morphisms, and products are again prevarieties.

Construction 5.12 (Open and closed subprevarieties)**.** Let X be a prevariety.

(a) Let $U \subset X$ be an open subset. Then U is again a prevariety (as usual with the structure sheaf $\mathscr{O}_U = \mathscr{O}_X|_U$ as in Definition 4.1 (c)): as X can be covered by affine varieties, U can be covered by open subsets of affine varieties, which themselves can be covered by affine varieties by Example 5.3.

We call U (with this structure as a prevariety) an **open subprevariety** of X.

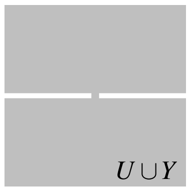
(b) The situation is more complicated for a closed subset $Y \subset X$: as an open subset U of Y is in general not open in X we cannot define a structure sheaf on Y by simply setting $\mathscr{O}_Y(U)$ to be $\mathscr{O}_X(U)$. Instead, we define $\mathscr{O}_Y(U)$ to be the K-algebra of functions $U \to K$ that are locally restrictions of functions on X, or formally

$$\mathscr{O}_Y(U) := \{\varphi : U \to K : \text{for all } a \in U \text{ there are an open neighborhood } V \text{ of } a \text{ in } X \\ \text{and } \psi \in \mathscr{O}_X(V) \text{ with } \varphi = \psi \text{ on } U \cap V\}.$$

By the local nature of this definition it is obvious that $\mathscr{O}_Y$ is a sheaf, thus making Y into a ringed space. In fact, we will check in Exercise 5.13 that Y is a prevariety in this way. We call it a **closed subprevariety** of X.

(c) If U is an open and Y a closed subset of X, then $U \cap Y$ is open in Y and closed in U, and thus we can give it the structure of a prevariety by combining (a) with (b) — in fact, one can check that it does not matter whether we consider it to be an open subprevariety of Y or a closed subprevariety of U. Intersections of open and closed subprevarieties (with this structure of a ringed space) are called **locally closed subprevarieties**. For example, $\{(x_1, x_2) \in \mathbb{A}^2 : x_1 = 0, x_2 \neq 0\}$ is a locally closed subprevariety of $\mathbb{A}^2$.

The reason why we consider all these seemingly special cases is that for a general subset of X there is no way to make it into a prevariety in a natural way. Even worse, the notions of open and closed subprevarieties do not mix very well: whereas a finite union of open (resp. closed) subprevarieties is of course again an open (resp. closed) subprevariety, the same statement is not true if we try to combine open with closed spaces: in $X = \mathbb{A}^2$ the union of the open subprevariety $U = \mathbb{A}^1 \times (\mathbb{A}^1 \backslash \{0\})$ and the closed subprevariety $Y = \{0\}$ as in the picture on the right does not have a natural structure as a subprevariety of $\mathbb{A}^2$ (since it does not look like an affine variety in a neighborhood of the origin).



Exercise 5.13. Let Y be a closed subset of a prevariety X, considered as a ringed space with the structure sheaf of Construction 5.12 (b). Prove for every affine open subset $U \subset X$ that the ringed space $U \cap Y$ (considered as an open subset of the ringed space Y as in Definition 4.1 (c)) is isomorphic to the affine variety $U \cap Y$ (considered as an affine subvariety of the affine variety U).

In particular, this shows that Construction 5.12 (b) makes Y into a prevariety, and that this prevariety is isomorphic to the affine variety Y if X is itself affine (and thus Y an affine subvariety of X).

Remark 5.14 (Properties of closed subprevarieties)**.** By Construction 5.12 (b), a regular function on (an open subset of) a closed subprevariety Y of a prevariety X is locally the restriction of a regular function on X. Hence:

(a) The inclusion map $Y \to X$ is a morphism (it is clearly continuous, and regular functions on X are by construction still regular when restricted to Y).

(b) If $f : Z \to X$ is a morphism from a prevariety Z such that $f(Z) \subset Y$ then we can also regard f as a morphism from Z to Y (the pull-back of a regular function on Y by f is locally also a pull-back of a regular function on X, and hence regular as $f : Z \to X$ is a morphism).

Remark 5.15 (Images and inverse images of subprevarieties)**.** Let $f : X \to Y$ be a morphism of prevarieties.

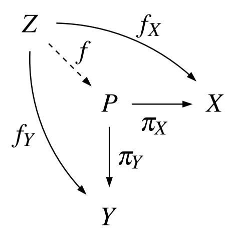
(a) The image of an open or closed subprevariety of X under f is not necessarily an open or closed subprevariety of Y. For example, for the affine variety $X = V(x_2x_3 - 1) \cup \{0\} \subset \mathbb{A}^3$ and the projection morphism $f : X \to \mathbb{A}^2$ onto the first two coordinates the image $f(X)$ is exactly the space $\mathbb{A}^1 \times (\mathbb{A}^1 \backslash \{0\}) \cup \{0\}$ of Construction 5.12 which is neither an open nor a closed subprevariety of $\mathbb{A}^2$.

As a substitute, one can often consider the graph of f instead of its image, see Proposition 5.21 (a).

(b) By continuity, the inverse image of an open (resp. closed) subprevariety of Y under f is clearly again an open (resp. closed) subprevariety of X.

As for the product $X \times Y$ of two prevarieties X and Y, the natural idea to construct this space as a prevariety would be to choose finite affine open covers $\{U_i : i \in I\}$ and $\{V_j : j \in J\}$ of X and Y, respectively, and then glue the affine product varieties $U_i \times V_j$ using Construction 5.7. If we did this directly however, we would still have to prove that the resulting space does not depend on the chosen affine covers. The best way out of this trouble is to define the product prevariety by a universal property analogous to Proposition 4.10. This will then ensure the uniqueness of the product, so that it suffices to prove its existence by gluing affine patches.

Definition 5.16 (Products of prevarieties)**.** Let X and Y be prevarieties. A **product** of X and Y is a prevariety P together with morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$ satisfying the following *universal property*: for any two morphisms $f_X : Z \to X$ and $f_Y : Z \to Y$ from another prevariety Z there is a unique morphism $f : Z \to P$ such that $\pi_X \circ f = f_X$ and $\pi_Y \circ f = f_Y$.



As in the affine case in Proposition 4.10, this means that giving a morphism to the product P is the same as giving a morphism to each of the factors X and Y.

Proposition 5.17 (Existence and uniqueness of products)**.** *Any two prevarieties X and Y have a product. Moreover, this product P with morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$ is unique up to unique isomorphism: if P' with $\pi'_X : P' \to X$ and $\pi'_Y : P' \to Y$ is another product there is a unique isomorphism $f : P' \to P$ such that $\pi_X \circ f = \pi'_X$ and $\pi_Y \circ f = \pi'_Y$.*

We will denote this product simply by $X \times Y$.

Proof. To show existence we glue the affine products of Proposition 4.10 using Construction 5.7. More precisely, let X and Y be covered by affine varieties U_i and V_j for $i \in I$ and $j \in J$, respectively. Use Construction 5.7 to glue the affine products $U_i \times V_j$, where we glue any two such products $U_i \times V_j$ and $U_{i'} \times V_{j'}$ along the identity isomorphism of the common open subset $(U_i \cap U_{i'}) \times (V_j \cap V_{j'})$. Note that these isomorphisms obviously satisfy the conditions (a) and (b) of the construction, and that the resulting glued space P is just the usual product $X \times Y$ as a set. Moreover, using Lemma 4.6 we can glue the affine projection morphisms $U_i \times V_j \to U_i \subset X$ and $U_i \times V_j \to V_j \subset Y$ to morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$.

Let us now check the universal property of Definition 5.16 for our construction. If $f_X : Z \to X$ and $f_Y : Z \to Y$ are any two morphisms from a prevariety Z, the only way to achieve $\pi_X \circ f = f_X$ and $\pi_Y \circ f = f_Y$ is to define $f : Z \to P$ as $f(z) = (f_X(z), f_Y(z))$, where we identify P set-theoretically with $X \times Y$. By Lemma 4.6, we can check that this is a morphism by restricting it to an affine open cover. If we first cover Z by the open subsets $f_X^{-1}(U_i) \cap f_Y^{-1}(V_j)$ for all $i \in I$ and $j \in J$, and these subsets then by affine open subsets by Construction 5.12 (a), we may assume that every affine open subset in our open cover of Z is mapped to a single (and hence affine) patch $U_i \times V_j$. But after this restriction to the affine case we know by Proposition 4.10 that f is a morphism.

To show uniqueness, assume that P' with $\pi'_X : P' \to X$ and $\pi'_Y : P' \to Y$ is another product. By the universal property of P applied to the morphisms $\pi'_X : P' \to X$ and $\pi'_Y : P' \to Y$, we see that there is a unique morphism $f : P' \to P$ with $\pi_X \circ f = \pi'_X$ and $\pi_Y \circ f = \pi'_Y$. Reversing the roles of the two products, we get in the same way a unique morphism $g : P \to P'$ with $\pi'_X \circ g = \pi_X$ and $\pi'_Y \circ g = \pi_Y$.

Finally, apply the universal property of P to the two morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$. Since both

$$\begin{aligned} \pi_X \circ (f \circ g) &= \pi'_X \circ g = \pi_X \\ \pi_Y \circ (f \circ g) &= \pi'_Y \circ g = \pi_Y \end{aligned} \qquad \text{and} \qquad \begin{aligned} \pi_X \circ \mathrm{id}_P &= \pi_X \\ \pi_Y \circ \mathrm{id}_P &= \pi_Y \end{aligned}$$

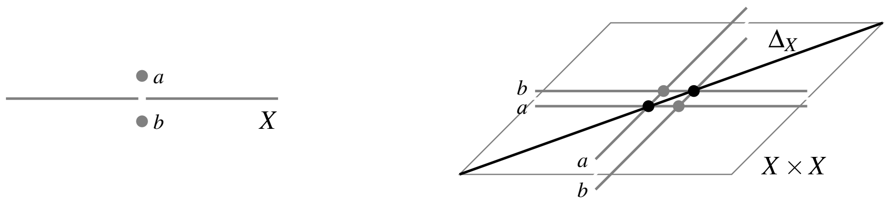
the uniqueness part of the universal property shows that $f \circ g = \mathrm{id}_P$. In the same way we see that $g \circ f = \mathrm{id}_{P'}$, so that f is an isomorphism. □

Remark 5.18. Again, a check might be in order that our constructions were consistent: let X and Y be prevarieties, and let $X' \subset X$ and $Y' \subset Y$ be closed subprevarieties. Then we have defined two structures of prevarieties on the set-theoretic product $X' \times Y'$: the closed subprevariety structure of $X' \times Y'$ in $X \times Y$ as in Construction 5.12 (b), and the product prevariety structure of Definition 5.16. As expected, these two structures agree: in fact, by Definition 5.16 together with Remark 5.14 the set-theoretic identity map is a morphism between these two structures in both ways.

Finally, as already announced let us now impose a condition on prevarieties that excludes such unwanted spaces as the affine line with two zero points of Example 5.5 (b). In the theory of manifolds, this is usually done by requiring that the topological space satisfies the so-called *Hausdorff property*, i. e. that every two distinct points have disjoint open neighborhoods. This is obviously not satisfied in our case, since the two zero points do not have such disjoint open neighborhoods.

However, in the Zariski topology the Hausdorff property does not make too much sense, as non-empty open subsets of an irreducible space can never be disjoint by Remark 2.18 (a). So we need a suitable replacement for this condition that captures our geometric idea of the absence of doubled points also in the Zariski topology.

The solution to this problem is inspired by a proposition in general topology stating that the Hausdorff property of a topological space X is equivalent to the condition that the so-called *diagonal* $\Delta_X = \{(x,x) : x \in X\}$ is closed in $X \times X$ (with the product topology). The picture below illustrates this in the case when X is the affine line with two zero points a and b: the product $X \times X$ then contains four zero points (a,a), (a,b), (b,a), and (b,b), but by definition only two of them, namely (a,a) and (b,b), are in Δ_X. Hence the diagonal is not closed: the other two zero points are also contained in its closure.



Of course, this equivalence does not really help us directly in algebraic geometry since we do not use the product topology on $X \times X$. But the geometric idea to detect doubled points shown in the picture above on the right is still valid in the Zariski topology — and so we will just use the diagonal condition to *define* the property that we want:

Definition 5.19 (Varieties)**.**

(a) A prevariety X is called a **variety** (or **separated**) if the **diagonal**

$$\Delta_X := \{(x,x) : x \in X\}$$

is closed in $X \times X$.

(b) Analogously to Definition 2.30 (b), a variety of pure dimension 1 or 2 is called a **curve** resp. **surface**. If X is a pure-dimensional variety and Y a pure-dimensional subvariety of X with $\dim Y = \dim X - 1$ we say that Y is a **hypersurface** in X.

So by the argument given above, the affine line with two zero points of Example 5.5 (b) is not a variety. In contrast, the following lemma shows that most prevarieties that we will meet are also varieties. From now on we will almost always assume that our spaces are separated, and thus talk about varieties instead of prevarieties.

Lemma 5.20.

(a) *Affine varieties are varieties.*

(b) *Open, closed, and locally closed subprevarieties of varieties are varieties. We will therefore simply call them* ***open***, ***closed***, *and* ***locally closed subvarieties***, *respectively.*

Proof.

(a) If $X \subset \mathbb{A}^n$ then $\Delta_X = V(x_1 - y_1, \dots, x_n - y_n) \subset X \times X$, where $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are the coordinates on the two factors, respectively. Hence Δ_X is closed.

(b) If $Y \subset X$ is an open, closed, or locally closed subvariety, consider the inclusion morphism $i: Y \times Y \to X \times X$ (which exists by the universal property of Definition 5.16). As $\Delta_Y = i^{-1}(\Delta_X)$ and Δ_X is closed by assumption, Δ_Y is closed as well by the continuity of i. □

For varieties, we have the following additional desirable properties in addition to the ones for prevarieties:

Proposition 5.21 (Properties of varieties)**.** *Let $f, g : X \to Y$ be morphisms of prevarieties, and assume that Y is a variety.*

(a) *The* ***graph*** $\Gamma_f := \{(x, f(x)) : x \in X\}$ *is closed in* $X \times Y$.

(b) *The set* $\{x \in X : f(x) = g(x)\}$ *is closed in* X.

Proof.

(a) By the universal property of products of prevarieties as in Definition 5.16 there is a morphism $(f, \mathrm{id}) : X \times Y \to Y \times Y$, $(x, y) \mapsto (f(x), y)$. As Y is a variety we know that Δ_Y is closed, and hence so is $\Gamma_f = (f, \mathrm{id})^{-1}(\Delta_Y)$ by continuity.

(b) Similarly to (a), the given set is the inverse image of the diagonal Δ_Y under the morphism $X \to Y \times Y$, $x \mapsto (f(x), g(x))$. Hence it is closed again since Δ_Y is closed. □

Exercise 5.22. Show that the space $\mathbb{P}^1$ of Example 5.5 (a) is a variety.

Exercise 5.23. Let X and Y be prevarieties. Show:

(a) If X and Y are varieties then so is $X \times Y$.

(b) If X and Y are irreducible then so is $X \times Y$.

Exercise 5.24. Use diagonals to prove the following statements:

(a) The intersection of any two affine open subsets of a variety is again an affine open subset.

(b) If $X, Y \subset \mathbb{A}^n$ are two pure-dimensional affine varieties then every irreducible component of $X \cap Y$ has dimension at least $\dim X + \dim Y - n$.

Exercise 5.25. In Exercise 2.33 (b) we have seen that the dimension of a dense open subset U of a topological space X need not be the same as that of X.

However, show now that $\dim U = \dim X$ holds in this situation if X is a variety. 08