3. The Sheaf of Regular Functions

After having defined affine varieties, our next goal must be to say what kind of maps between them we want to consider as morphisms, i. e. as "nice maps preserving the structure of the variety". In this chapter we will look at the easiest case of this: the so-called *regular functions*, i. e. maps to the ground field $K = \mathbb{A}^1$. They should be thought of as the analogue of continuous functions in topology, differentiable functions in real analysis, or holomorphic functions in complex analysis.

So what kind of nice "algebraic" functions should we consider on an affine variety X? First of all, as in the case of continuous or differentiable functions, we should not only aim for a definition of functions on all of X, but also on an arbitrary open subset U of X. In contrast to the coordinate ring $A(X)$ of polynomial functions on the whole space X, this allows us to consider quotients $\frac{g}{f}$ of polynomial functions $f, g \in A(X)$ with $f \neq 0$ as well, since we can exclude the zero set $V(f)$ of the denominator from the domain of definition of the function.

But taking our functions to be quotients of polynomials turns out to be a little bit too restrictive. The problem with this definition would be that *it is not local*: recall that the condition on a function to be continuous or differentiable is local in the sense that it can be checked at every point, with the whole function then being continuous or differentiable if it has this property at every point. Being a quotient of polynomials however is not a condition of this type — we would have to find one *global* representation as a quotient of polynomials that is then valid at every point. Imposing such non-local conditions is usually not a good thing to do, since it would be hard in practice to find the required global representations of the functions.

The way out of this problem is to consider functions that are only *locally* quotients of polynomials, i. e. functions $\varphi : U \to K$ such that each point $a \in U$ has a neighborhood in U in which $\varphi = \frac{g}{f}$ holds for two polynomials f and g (that may depend on a). In fact, we will see in Example 3.5 that passing from *global* to *local* quotients of polynomials really makes a difference. So let us now give the corresponding formal definition of regular functions.

Definition 3.1 (Regular functions)**.** Let X be an affine variety, and let U be an open subset of X. A **regular function** on U is a map $\varphi : U \to K$ with the following property: for every $a \in U$ there are polynomial functions $f, g \in A(X)$ with $f(x) \neq 0$ and

$$\varphi(x) = \frac{g(x)}{f(x)}$$

for all x in an open subset U_a with $a \in U_a \subset U$. The set of all such regular functions on U will be denoted $\mathscr{O}_X(U)$.

Notation 3.2. We will usually write the condition "$\varphi(x) = \frac{g(x)}{f(x)}$ for all $x \in U_a$" of Definition 3.1 simply as "$\varphi = \frac{g}{f}$ on U_a". This is certainly an intuitive notation that should not lead to any confusion. However, a word of warning in particular for those of you who know commutative algebra already: this also means that (unless stated otherwise) the fraction $\frac{g}{f}$ of two elements of $A(X)$ will always denote the pointwise quotient of the two corresponding polynomial functions — and not the algebraic concept of an element in a localized ring as introduced later in Construction 3.12.

Remark 3.3 ($\mathscr{O}_X(U)$ as a ring and K-algebra)**.** It is obvious that the set $\mathscr{O}_X(U)$ of regular functions on an open subset U of an affine variety X is a ring with pointwise addition and multiplication. However, it has an additional structure: it is also a K-vector space since we can multiply a regular function pointwise with a fixed scalar in K. In algebraic terms, this means that $\mathscr{O}_X(U)$ is a K-algebra, which is defined as follows.

Definition 3.4 (K-algebras [G5, Definition 1.23 and Remark 1.24])**.**

(a) A K**-algebra** is a ring R that is at the same time a K-vector space such that the ring multiplication is K-bilinear.

(b) For two K-algebras R and R' a **morphism** (or K**-algebra homomorphism**) from R to R' is a map $f : R \to R'$ that is a ring homomorphism as well as a K-linear map.

Example 3.5 (Local $\neq$ global quotients of polynomials)**.** Consider the 3-dimensional affine variety $X = V(x_1x_4 - x_2x_3) \subset \mathbb{A}^4$ and the open subset

$$U = X\backslash V(x_2,x_4) = \{(x_1,x_2,x_3,x_4) \in X : x_2 \neq 0 \text{ or } x_4 \neq 0\} \quad \subset X.$$

Then

$$\varphi : U \to K, \ (x_1,x_2,x_3,x_4) \mapsto \begin{cases} \frac{x_1}{x_2} & \text{if } x_2 \neq 0, \\ \frac{x_3}{x_4} & \text{if } x_4 \neq 0 \end{cases} \qquad (*)$$

is a regular function on U: it is well-defined since the defining equation for X implies $\frac{x_1}{x_2} = \frac{x_3}{x_4}$ whenever $x_2 \neq 0$ and $x_4 \neq 0$, and it is obviously locally a quotient of polynomials. But none of the two representations in $(*)$ as quotients of polynomials can be used on all of U, since the first one does not work e. g. at the point $(0,0,0,1) \in U$, whereas the second one does not work at $(0,1,0,0) \in U$. Algebraically, this is just the statement that $A(X)$ is not a unique factorization domain [G5, Definition 8.1] because of the relation $x_1x_4 = x_2x_3$.

In fact, one can show that there is also no other global representation of φ as a quotient of two polynomials. We will not need this statement here, and so we do not prove it — we should just keep in mind that representations of regular functions as quotients of polynomials will in general not be valid on the complete domain of definition of the function.

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As a first result, let us prove the expected statement that zero loci of regular functions are always closed in their domain of definition.

Lemma 3.6 (Zero loci of regular functions are closed)**.** *Let U be an open subset of an affine variety X, and let $\varphi \in \mathscr{O}_X(U)$ be a regular function on U. Then*

$$V(\varphi) := \{x \in U : \varphi(x) = 0\}$$

is closed in U.

Proof. By Definition 3.1 any point $a \in U$ has an open neighborhood U_a in U on which $\varphi = \frac{g_a}{f_a}$ for some $f_a, g_a \in A(X)$ (with f_a nowhere zero on U_a). So the set

$$\{x \in U_a : \varphi(x) \neq 0\} = U_a \backslash V(g_a)$$

is open in X, and hence so is their union over all $a \in U$, which is just $U\backslash V(\varphi)$. This means that $V(\varphi)$ is closed in U. □

Remark 3.7 (Identity Theorem for regular functions)**.** A simple but remarkable consequence of Lemma 3.6 is the following: let $U \subset V$ be non-empty open subsets of an irreducible affine variety X. If $\varphi_1, \varphi_2 \in \mathscr{O}_X(V)$ are two regular functions on V that agree on U, then they must agree on all of V: the locus $V(\varphi_1 - \varphi_2)$ where the two functions agree contains U and is closed in V by Lemma 3.6, hence it also contains the closure $\overline{U}$ of U in V. But $\overline{V} = X$ by Remark 2.18 (b), hence V is irreducible by Exercise 2.19 (b), which again by Remark 2.18 (b) means that the closure of U in V is V. Consequently, we have $\varphi_1 = \varphi_2$ on V.

Note that this statement is not really surprising since the open subsets in the Zariski topology are so big: over the ground field $\mathbb{C}$, for example, it is also true in the classical topology that the closure of U in V is V, and hence the equation $\varphi_1 = \varphi_2$ on V already follows from $\varphi_1|_U = \varphi_2|_U$ by the (classical) continuity of φ_1 and φ_2. The interesting fact here is that the very same statement holds in complex analysis for holomorphic functions as well (or more generally, in real analysis for analytic functions): two holomorphic functions on a (connected) open subset $U \subset \mathbb{C}^n$ must be the same if they agree on any smaller open subset $V \subset U$. This is called the *Identity Theorem* for holomorphic functions. In complex analysis this is a real theorem because the open subset V can be very small, so the statement that the extension to U is unique is a lot more surprising than it is here in algebraic geometry. Still this is an example of a theorem that is true in literally the same way in both algebraic and complex geometry, although these two theories are quite different a priori. We will see another case of this in Example 3.14.

Let us now go ahead and compute the K-algebras $\mathscr{O}_X(U)$ in some cases. A particularly important result in this direction can be obtained if U is the complement of the zero locus of a single polynomial function $f \in A(X)$. In this case it turns out that (in contrast to Example 3.5) the regular functions on U can always be written with a single representation as a fraction, whose denominator is in addition a power of f.

Definition 3.8 (Distinguished open subsets)**.** For an affine variety X and a polynomial function $f \in A(X)$ on X we call

$$D(f) := X \backslash V(f) = \{x \in X : f(x) \neq 0\}$$

the **distinguished open subset** of f in X.

Remark 3.9. The distinguished open subsets of an affine variety X behave nicely with respect to intersections and unions:

(a) For any $f, g \in A(X)$ we have $D(f) \cap D(g) = D(fg)$, since $f(x) \neq 0$ and $g(x) \neq 0$ is equivalent to $(fg)(x) \neq 0$ for all $x \in X$. In particular, finite intersections of distinguished open subsets are again distinguished open subsets.

(b) Any open subset $U \subset X$ is a finite union of distinguished open subsets: by definition of the Zariski topology it is the complement of an affine variety, which in turn is the zero locus of finitely many polynomial functions $f_1, \dots, f_k \in A(X)$ by Proposition 1.21 (a). Hence we have

$$U = X \backslash V(f_1, \dots, f_k) = D(f_1) \cup \dots \cup D(f_k).$$

We can therefore think of the distinguished open subsets as the "smallest" open subsets of X — in topology, the correct notion for this would be to say that they form a *basis* of the Zariski topology on X.

Proposition 3.10 (Regular functions on distinguished open subsets)**.** *Let X be an affine variety, and let $f \in A(X)$. Then*

$$\mathscr{O}_X(D(f)) = \left\{ \frac{g}{f^n} : g \in A(X), n \in \mathbb{N} \right\}.$$

In particular, setting $f = 1$ we see that $\mathscr{O}_X(X) = A(X)$, i. e. the regular functions on all of X are exactly the polynomial functions.

Proof. The inclusion "$\supset$" is obvious: every function of the form $\frac{g}{f^n}$ for $g \in A(X)$ and $n \in \mathbb{N}$ is clearly regular on $D(f)$.

For the opposite inclusion "$\subset$", let $\varphi : D(f) \to K$ be a regular function. By Definition 3.1 we obtain for every $a \in D(f)$ a local representation $\varphi = \frac{g_a}{f_a}$ for some $f_a, g_a \in A(X)$ which is valid on an open neighborhood of a in $D(f)$. After possibly shrinking these neighborhoods we may assume by Remark 3.9 (b) that they are distinguished open subsets $D(h_a)$ for some $h_a \in A(X)$. Moreover, we can change the representations of φ by replacing g_a and f_a by $g_a h_a$ and $f_a h_a$ (which does not change their quotient on $D(h_a)$) to assume that both g_a and f_a vanish on the complement $V(h_a)$ of $D(h_a)$. Finally, this means that f_a vanishes on $V(h_a)$ and does not vanish on $D(h_a)$ — so h_a and f_a have the same zero locus, and we can therefore assume that $h_a = f_a$.

As a consequence, note that

$$g_a f_b = g_b f_a \quad \text{for all } a, b \in D(f) : \qquad (*)$$

these two functions agree on $D(f_a) \cap D(f_b)$ since $\varphi = \frac{g_a}{f_a} = \frac{g_b}{f_b}$ there, and they are both zero otherwise since by our construction we have $g_a(x) = f_a(x) = 0$ for all $x \in V(f_a)$ and $g_b(x) = f_b(x) = 0$ for all $x \in V(f_b)$.

Now all our open neighborhoods cover $D(f)$, i.e. we have $D(f) = \bigcup_{a\in D(f)} D(f_a)$. Passing to the complement we obtain

$$V(f) = \bigcap_{a\in D(f)} V(f_a) = V\big(\{f_a : a \in D(f)\}\big),$$

and thus by Proposition 1.21 (b)

$$f \in I(V(f)) = I\big(V\big(\{f_a : a \in D(f)\}\big)\big) = \sqrt{(f_a : a \in D(f))}.$$

This means that $f^n = \sum_a k_a f_a$ for some $n \in \mathbb{N}$ and $k_a \in A(X)$ for finitely many elements $a \in D(f)$. Setting $g := \sum_a k_a g_a$, we then claim that $\varphi = \frac{g}{f^n}$ on all of $D(f)$: for all $b \in D(f)$ we have $\varphi = \frac{g_b}{f_b}$ and

$$g f_b = \sum_a k_a g_a f_b \overset{(*)}{=} \sum_a k_a g_b f_a = g_b f^n$$

on $D(f_b)$, and these open subsets cover $D(f)$. □

Remark 3.11. In the proof of Proposition 3.10 we had to use Hilbert's Nullstellensatz again. In fact, the statement is false if the ground field is not algebraically closed, as you can see from the example of the function $\frac{1}{x^2+1}$ that is regular on all of $\mathbb{A}^1_{\mathbb{R}}$, but not a polynomial function.

Proposition 3.10 is deeply linked to commutative algebra. Although we considered the quotients $\frac{g}{f^n}$ in this statement to be fractions of polynomial functions, there is also a purely algebraic construction of fractions in a (polynomial) ring — in the same way as we could regard the elements of $A(X)$ either geometrically as functions on X or algebraically as elements in the quotient ring $K[x_1,\dots,x_n]/I(X)$. In these notes we will mainly use the geometric interpretation as functions, but it is still instructive to see the corresponding algebraic construction. For reasons that will become apparent in Lemma 3.21 it is called *localization*; it is also one of the central topics in the "Commutative Algebra" class.

Construction 3.12 (Localizations [G5, Chapter 6])**.** Let R be a ring. A **multiplicatively closed subset** of R is a subset $S \subset R$ with $1 \in S$ and $fg \in S$ for all $f,g \in S$.

For such a multiplicatively closed subset S we then consider pairs (g,f) with $g \in R$ and $f \in S$, and call two such pairs (g,f) and (g',f') equivalent if there is an element $h \in S$ with $h(gf' - g'f) = 0$. The equivalence class of a pair (g,f) will formally be written as a fraction $\frac{g}{f}$, the set of all such equivalence classes is denoted $S^{-1}R$. Together with the usual rules for the addition and multiplication of fractions, $S^{-1}R$ is again a ring. It is called the **localization** of R at S.

By construction we can think of the elements of $S^{-1}R$ as formal fractions, with the numerators in R and the denominators in S. In Proposition 3.10 our set of denominators is $S = \{f^n : n \in \mathbb{N}\}$ for a fixed element $f \in R$; in this case the localization $S^{-1}R$ is usually written as R_f. We will meet other sets of denominators later in Lemma 3.21 and Exercise 9.8 (a).

So let us now prove rigorously that the K-algebra $\mathscr{O}_X(D(f))$ of Proposition 3.10 can also be interpreted algebraically as a localization.

Lemma 3.13 (Regular functions as localizations)**.** *Let X be an affine variety, and let $f \in A(X)$. Then $\mathscr{O}_X(D(f))$ is isomorphic (as a K-algebra) to the localized ring $A(X)_f$.*

Proof. There is an obvious K-algebra homomorphism

$$A(X)_f \to \mathscr{O}_X(D(f)), \quad \frac{g}{f^n} \mapsto \frac{g}{f^n}$$

that interprets a formal fraction in the localization $A(X)_f$ as an actual quotient of polynomial functions on $D(f)$. It is in fact well-defined: if $\frac{g}{f^n} = \frac{g'}{f^m}$ as formal fractions in the localization $A(X)_f$ then $f^k(gf^m - g'f^n) = 0$ in $A(X)$ for some $k \in \mathbb{N}$, which means that $gf^m = g'f^n$ and thus $\frac{g}{f^n} = \frac{g'}{f^m}$ as functions on $D(f)$.

The homomorphism is surjective by Proposition 3.10. It is also injective: if $\frac{g}{f^n} = 0$ as a function on $D(f)$ then $g = 0$ on $D(f)$ and hence $fg = 0$ on all of X, which means $f(g\cdot 1 - 0\cdot f^n) = 0$ in $A(X)$ and thus $\frac{g}{f^n} = \frac{0}{1}$ as formal fractions in the localization $A(X)_f$. □

Example 3.14 (Regular functions on $\mathbb{A}^2\backslash\{0\}$)**.** Probably the easiest case of an open subset of an affine variety that is not a distinguished open subset is the complement $U = \mathbb{A}^2\backslash\{0\}$ of the origin in the affine plane $X = \mathbb{A}^2$. We are going to see that

$$\mathscr{O}_{\mathbb{A}^2}(\mathbb{A}^2\backslash\{0\}) = K[x_1,x_2]$$

and thus that $\mathscr{O}_X(U) = \mathscr{O}_X(X)$, i. e. every regular function on U can be extended to X. Note that this is another result that is true in the same way in complex analysis: there is a *Removable Singularity Theorem* that implies that every holomorphic function on $\mathbb{C}^2\backslash\{0\}$ can be extended holomorphically to $\mathbb{C}^2$.

To prove our claim let $\varphi \in \mathscr{O}_X(U)$. Then φ is regular on the distinguished open subsets $D(x_1) = (\mathbb{A}^1\backslash\{0\}) \times \mathbb{A}^1$ and $D(x_2) = \mathbb{A}^1 \times (\mathbb{A}^1\backslash\{0\})$, and so by Proposition 3.10 we can write $\varphi = \frac{f}{x_1^m}$ on $D(x_1)$ and $\varphi = \frac{g}{x_2^n}$ on $D(x_2)$ for some $f,g \in K[x_1,x_2]$ and $m,n \in \mathbb{N}$. Of course we can do this so that $x_1 \nmid f$ and $x_2 \nmid g$.

On the intersection $D(x_1) \cap D(x_2)$ both representations of φ are valid, and so we have $f x_2^n = g x_1^m$ on $D(x_1) \cap D(x_2)$. But the locus $V(f x_2^n - g x_1^m)$ where this equation holds is closed, and hence we see that $f x_2^n = g x_1^m$ also on $\overline{D(x_1) \cap D(x_2)} = \mathbb{A}^2$. In other words, we have $f x_2^n = g x_1^m$ in the polynomial ring $A(\mathbb{A}^2) = K[x_1,x_2]$.

Now if we had $m > 0$ then x_1 must divide $f x_2^n$, which is clearly only possible if $x_1 \,|\, f$. This is a contradiction, and so it follows that $m = 0$. But then $\varphi = f$ is a polynomial, as we have claimed.

Exercise 3.15. For those of you who know some commutative algebra already: generalize the proof of Example 3.14 to show that $\mathscr{O}_X(U) = \mathscr{O}_X(X) = A(X)$ for any open subset U of an affine variety X such that $A(X)$ is a unique factorization domain [G5, Definition 8.1] and U is the complement of an irreducible subvariety of codimension at least 2 in X.

Recall that we have defined regular functions on an open subset U of an affine variety as set-theoretic functions from U to the ground field K that satisfy some local property. Local constructions of function-like objects occur in many places in algebraic geometry (and also in other "topological" fields of mathematics), and so we will spend the rest of this chapter to formalize the idea of such objects. This will have the advantage that it gives us an "automatic" definition of morphisms between affine varieties in Chapter 4, and in fact also between more general varieties in Chapter 5.

Definition 3.16 (Sheaves)**.** A **presheaf** $\mathscr{F}$ (of rings) on a topological space X consists of the data:

- for every open set $U \subset X$ a ring $\mathscr{F}(U)$ (think of this as the ring of functions on U),
- for every inclusion $U \subset V$ of open sets in X a ring homomorphism $\rho_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$ called the *restriction map* (think of this as the usual restriction of functions to a subset),

such that

- $\mathscr{F}(\emptyset) = 0$,
- $\rho_{U,U}$ is the identity map on $\mathscr{F}(U)$ for all U,
- for any inclusion $U \subset V \subset W$ of open sets in X we have $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$.

The elements of $\mathscr{F}(U)$ are usually called the **sections** of $\mathscr{F}$ over U, and the restriction maps $\rho_{V,U}$ are written as $\varphi \mapsto \varphi|_U$.

A presheaf $\mathscr{F}$ is called a **sheaf** of rings if it satisfies the following gluing property: if $U \subset X$ is an open set, $\{U_i : i \in I\}$ an arbitrary open cover of U and $\varphi_i \in \mathscr{F}(U_i)$ sections for all i such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ for all $i,j \in I$, then there is a unique $\varphi \in \mathscr{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all i.

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Example 3.17. Intuitively speaking, any "function-like" object forms a presheaf; it is a sheaf if the conditions imposed on the "functions" are local (i. e. if they can be checked on an open cover). The following examples illustrate this.

(a) Let X be an affine variety. Then the rings $\mathscr{O}_X(U)$ of regular functions on open subsets $U \subset X$, together with the usual restriction maps of functions, form a sheaf $\mathscr{O}_X$ on X. In fact, the presheaf axioms are obvious, and the gluing property just means that a function $\varphi : U \to K$ is regular if it is regular on each element of an open cover of U (which follows from the definition that φ is regular if it is *locally* a quotient of polynomial functions). We call $\mathscr{O}_X$ the *sheaf of regular functions* on X.

(b) Similarly, on $X = \mathbb{R}^n$ the rings

$$\mathscr{F}(U) = \{\varphi : U \to \mathbb{R} \text{ continuous}\}$$

for open subsets $U \subset X$ form a sheaf $\mathscr{F}$ on X with the usual restriction maps. In the same way we can consider on X the sheaves of differentiable functions, analytic functions, arbitrary functions, and so on.

(c) On $X = \mathbb{R}^n$ let

$$\mathscr{F}(U) = \{\varphi : U \to \mathbb{R} \text{ constant function}\}$$

with the usual restriction maps. Then $\mathscr{F}$ is a presheaf, but not a sheaf, since being a constant function is not a local condition. More precisely, let U_1 and U_2 be any non-empty disjoint open subsets of X, and let $\varphi_1 \in \mathscr{F}(U_1)$ and $\varphi_2 \in \mathscr{F}(U_2)$ be constant functions with different values. Then φ_1 and φ_2 trivially agree on $U_1 \cap U_2 = \emptyset$, but there is still no *constant* function on $U = U_1 \cup U_2$ that restricts to both φ_1 on U_1 and φ_2 on U_2. Hence $\mathscr{F}$ does not satisfy the gluing property. Note however that we would obtain a sheaf if we considered *locally constant* functions instead of constant ones.

In order to get used to the language of sheaves let us now consider two common constructions with them.

Definition 3.18 (Restrictions of (pre-)sheaves)**.** Let $\mathscr{F}$ be a presheaf on a topological space X, and let $U \subset X$ be an open subset. Then the **restriction** of $\mathscr{F}$ to U is defined to be the presheaf $\mathscr{F}|_U$ on U with

$$\mathscr{F}|_U(V) := \mathscr{F}(V)$$

for every open subset $V \subset U$, and with the restriction maps taken from $\mathscr{F}$. Note that if $\mathscr{F}$ is a sheaf then so is $\mathscr{F}|_U$.

Construction 3.19 (Stalks of (pre-)sheaves)**.** Again let $\mathscr{F}$ be a presheaf on a topological space X. Fix a point $a \in X$ and consider pairs (U, φ) where U is an open neighborhood of a and $\varphi \in \mathscr{F}(U)$. We call two such pairs (U, φ) and (U', φ') equivalent if there is an open subset V with $a \in V \subset U \cap U'$ and $\varphi|_V = \varphi'|_V$ (it is easy to check that this is indeed an equivalence relation). The set of all such pairs modulo this equivalence relation is called the **stalk** $\mathscr{F}_a$ of $\mathscr{F}$ at a; it inherits a ring structure from the rings $\mathscr{F}(U)$. The elements of $\mathscr{F}_a$ are called **germs** of $\mathscr{F}$ at a.

Remark 3.20. The geometric interpretation of the germs of a sheaf is that they are functions (resp. sections of the sheaf) that are defined in an arbitrarily small neighborhood of the given point — we will also refer to these objects as *local functions* at this point. Hence e. g. on the real line the germ of a differentiable function at a point a allows you to compute the derivative of this function at a, but none of the actual values of the function at any point except a. On the other hand, we have seen in Remark 3.7 that holomorphic functions on a (connected) open subset of $\mathbb{C}^n$ are already determined by their values on any smaller open set. So in this sense germs of holomorphic functions carry “much more information” than germs of differentiable functions.

In algebraic geometry, the situation is similar: let φ_1 and φ_2 be two regular functions on an open subset U of an irreducible affine variety X. If there is a point of U at which the germs of φ_1 and φ_2 are the same then φ_1 and φ_2 have to agree on a non-empty open subset, which means by Remark 3.7 that $\varphi_1 = \varphi_2$ on U. In other words, the germ of a regular function determines the function uniquely already. Note that the corresponding statement is clearly false for differentiable functions as we have seen above.

In fact, germs of regular functions on an affine variety X can also be described algebraically in terms of localizations as introduced in Construction 3.12 — which is the reason why this algebraic concept is called "localization". As one might expect, such a germ at a point $a \in X$, i. e. a regular function in a small neighborhood of a, is given by an element in the localization of $A(X)$ for which we allow as denominators all polynomials that do not vanish at a.

Lemma 3.21 (Germs of regular functions as localizations)**.** *Let a be a point on an affine variety X, and let $S = \{f \in A(X) : f(a) \neq 0\}$. Then the stalk $\mathscr{O}_{X,a}$ of $\mathscr{O}_X$ at a is isomorphic (as a K-algebra) to the localized ring*

$$S^{-1}A(X) = \left\{ \frac{g}{f} : f, g \in A(X), f(a) \neq 0 \right\}.$$

It is called the ***local ring*** *of X at a.*

Proof. Note that S is clearly multiplicatively closed, so that the localization $S^{-1}A(X)$ exists. Consider the K-algebra homomorphism

$$S^{-1}A(X) \to \mathscr{O}_{X,a}, \quad \frac{g}{f} \mapsto \overline{\left(D(f), \frac{g}{f}\right)}$$

that maps a formal fraction $\frac{g}{f}$ to the corresponding quotient of polynomial functions on the open neighborhood of a where the denominator does not vanish. It is well-defined: if $\frac{g}{f} = \frac{g'}{f'}$ in the localization then $h(gf' - g'f) = 0$ for some $h \in S$. Hence the functions $\frac{g}{f}$ and $\frac{g'}{f'}$ agree on the open neighborhood $D(h)$ of a, and thus they determine the same element in the stalk $\mathscr{O}_{X,a}$.

The K-algebra homomorphism is surjective since by definition any regular function in a sufficiently small neighborhood of a must be representable by a fraction $\frac{g}{f}$ with $g \in A(X)$ and $f \in S$. It is also injective: assume that a function $\frac{g}{f}$ represents the zero element in the stalk $\mathscr{O}_{X,a}$, i. e. that it is zero in an open neighborhood of a. By possibly shrinking this neighborhood we may assume by Remark 3.9 (b) that it is a distinguished open subset $D(h)$ containing a, i. e. with $h \in S$. But then $h(g \cdot 1 - 0 \cdot f)$ is zero on all of X, hence zero in $A(X)$, and thus $\frac{g}{f} = \frac{0}{1}$ in the localization $S^{-1}A(X)$. □

Local rings will become important later on when we construct tangent spaces (see Lemma 10.5) and vanishing multiplicities (see Definition 12.23). We will then mostly use their algebraic description of Lemma 3.21 and write the elements of $\mathscr{O}_{X,a}$ as quotients $\frac{g}{f}$ with $f, g \in A(X)$ such that $f(a) \neq 0$.

Algebraically, the most important property of the local ring $\mathscr{O}_{X,a}$ is that it has only one *maximal ideal* in the sense of the following lemma. In fact, in commutative algebra a local ring is *defined* to be a ring with only one maximal ideal.

Lemma and Definition 3.22 (Maximal ideals)**.** *Let a be a point on an affine variety X. Then every proper ideal of the local ring $\mathscr{O}_{X,a}$ is contained in the ideal*

$$I_a := I(a)\,\mathscr{O}_{X,a} := \left\{ \frac{g}{f} : f, g \in A(X), g(a) = 0, f(a) \neq 0 \right\}$$

of all local functions vanishing at the point a. The ideal I_a is therefore called the ***maximal ideal*** *of $\mathscr{O}_{X,a}$.*

Proof. It is easily checked that I_a is in fact an ideal. Now let $I \trianglelefteq \mathscr{O}_{X,a}$ be any ideal not contained in I_a. By definition, this means that there is an element $\frac{g}{f} \in I$ with $f(a) \neq 0$ and $g(a) \neq 0$. But then $\frac{f}{g}$ exists in $\mathscr{O}_{X,a}$ as well. Hence $1 = \frac{f}{g} \cdot \frac{g}{f} \in I$, and we conclude that $I = \mathscr{O}_{X,a}$. □

Exercise 3.23. Let $X \subset \mathbb{A}^n$ be an affine variety, and let $a \in X$. Show that $\mathscr{O}_{X,a} \cong \mathscr{O}_{\mathbb{A}^n,a} / I(X)\,\mathscr{O}_{\mathbb{A}^n,a}$, where $I(X)\,\mathscr{O}_{\mathbb{A}^n,a}$ denotes the ideal in $\mathscr{O}_{\mathbb{A}^n,a}$ generated by all quotients $\frac{f}{1}$ for $f \in I(X)$.

Exercise 3.24. Let $\mathscr{F}$ be a sheaf on a topological space X, and let $a \in X$. Show that the stalk $\mathscr{F}_a$ is a local object in the following sense: if $U \subset X$ is an open neighborhood of a then $\mathscr{F}_a$ is isomorphic to the stalk of $\mathscr{F}|_U$ at a on the topological space U.

Remark 3.25 (Sheaves for other categories)**.** In Definition 3.16 we have constructed (pre-)sheaves of rings. In the same way one can define (pre-)sheaves of K-algebras, Abelian groups, or other suitable categories, by requiring that all $\mathscr{F}(U)$ are objects and all restriction maps are morphisms in the corresponding category. Note that the stalks of such a (pre-)sheaf then inherit this structure. For example, all our (pre-)sheaves considered so far have also been (pre-)sheaves of K-algebras for some field K, and thus their stalks are all K-algebras. In fact, starting in the next chapter we will restrict ourselves to this situation.