2. The Zariski Topology

In this chapter we will define a *topology* on an affine variety X, i. e. a notion of open and closed subsets of X. We will see that many properties of X can be expressed purely in terms of this topology, e. g. its dimension or the question whether it consists of several components. The advantage of this is that these concepts can then easily be reused later in Chapter 5 when we consider more general varieties — which are still topological spaces, but arise in a slightly different way.

Compared to e. g. the standard topology on subsets of real vector spaces, the properties of our topology on affine varieties will be very unusual. Consequently, most concepts and results covered in a standard introductory course on topology will be trivial or useless in our case, so that we will only need the very first definitions of general topology. Let us quickly review them here.

Remark 2.1 (Topologies)**.** A *topology* on a set X is given by declaring some subsets of X to be *closed*, such that the following properties hold:

(a) the empty set $\emptyset$ and the whole space X are closed;

(b) arbitrary intersections of closed sets are closed;

(c) finite unions of closed sets are closed.

Given such a topology on X, a subset U of X is then called *open* if its complement $X\backslash U$ is closed. The *closure* $\overline{A}$ of a subset $A \subset X$ is defined to be the smallest closed subset containing A, or more precisely the intersection of all closed subsets containing A (which is closed again by (b)).

A topology on X induces a *subspace topology* on any subset $A \subset X$ by declaring the subsets of A to be closed that are of the form $A \cap Y$ for a closed subset Y of X (or equivalently the subsets of A to be open that are of the form $A \cap U$ for an open subset U of X). Subsets of topological spaces will always be equipped with this subspace topology unless stated otherwise. Note that if A is closed itself then the closed subsets of A in the subspace topology are exactly the closed subsets of X contained in A; if A is open then the open subsets of A in the subspace topology are exactly the open subsets of X contained in A.

A map $\varphi : X \to Y$ between topological spaces is called *continuous* if inverse images of closed subsets of Y under φ are closed in X, or equivalently if inverse images of open subsets are open.

Note that the standard definition of closed subsets in $\mathbb{R}^n$ (or more generally in metric spaces) that you know from real analysis satisfies the conditions (a), (b), and (c), and leads with the above definitions to the well-known notions of open subsets, closures, and continuous functions.

With these preparations we can now define the standard topology used in algebraic geometry.

Definition 2.2 (Zariski topology)**.** Let X be an affine variety. We define the **Zariski topology** on X to be the topology whose closed sets are exactly the affine subvarieties of X, i. e. the subsets of the form $V(S)$ for some $S \subset A(X)$. Note that this in fact a topology by Example 1.4 (a) and Lemma 1.24.

Unless stated otherwise, topological notions for affine varieties (and their subsets, using the subspace topology of Remark 2.1) will always be understood with respect to this topology.

Remark 2.3. Let $X \subset \mathbb{A}^n$ be an affine variety. Then we have just defined two topologies on X:

(a) the Zariski topology on X, whose closed subsets are the affine subvarieties of X; and

(b) the subspace topology of X in $\mathbb{A}^n$, whose closed subsets are the sets of the form $X \cap Y$, with Y a variety in $\mathbb{A}^n$.

These two topologies agree, since the subvarieties of X are precisely the affine varieties contained in X and the intersection of two affine varieties is again an affine variety. Hence it will not lead to confusion if we consider both these topologies to be the standard topology on X.

Exercise 2.4. Let $X \subset \mathbb{A}^n$ be an arbitrary subset. Prove that $V(I(X)) = \overline{X}$.

Example 2.5 (Topologies on complex varieties)**.** Compared to the classical metric topology in the case of the ground field $\mathbb{C}$, the Zariski topology is certainly unusual:

(a) The metric unit ball $A = \{x \in \mathbb{A}^1_{\mathbb{C}} : |x| \leq 1\}$ in $\mathbb{A}^1_{\mathbb{C}}$ is clearly closed in the classical topology, but not in the Zariski topology. In fact, by Example 1.6 the Zariski-closed subsets of $\mathbb{A}^1$ are only the finite sets and $\mathbb{A}^1$ itself. In particular, the closure of A in the Zariski topology is all of $\mathbb{A}^1$.

Intuitively, we can say that the closed subsets in the Zariski topology are very "small", and hence that the open subsets are very "big" (see also Remark 2.18). Any Zariski-closed subset is also closed in the classical topology (since it is given by equations among polynomial functions, which are continuous in the classical topology), but as the above example shows only "very few" closed subsets in the classical topology are also Zariski-closed.

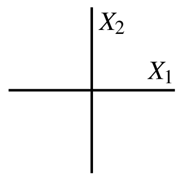
(b) Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ be *any* bijective map. Then φ is continuous in the Zariski topology, since inverse images of finite subsets of $\mathbb{A}^1$ under φ are finite.

This statement is essentially useless however, as we will not define morphisms of affine varieties as just being continuous maps with respect to the Zariski topology. In fact, this example gives us a strong hint that we should not do so.

(c) In general topology there is a notion of a *product topology*: if X and Y are topological spaces then $X \times Y$ has a natural structure of a topological space by saying that a subset of $X \times Y$ is open if and only if it is a union of products $U_i \times V_i$ for open subsets $U_i \subset X$ and $V_i \subset Y$ with i in an arbitrary index set.

With this construction, note however that the Zariski topology of an affine product variety $X \times Y$ is not the product topology: e. g. the subset $V(x_1 - x_2) = \{(a,a) : a \in K\} \subset \mathbb{A}^2$ is closed in the Zariski topology, but not in the product topology of $\mathbb{A}^1 \times \mathbb{A}^1$. In fact, we will see in Proposition 4.10 that the Zariski topology is the "correct" one, whereas the product topology is useless in algebraic geometry.

But let us now start with the discussion of the topological concepts that are actually useful in the Zariski topology. The first ones concern *components* of an affine variety: the affine variety $X = V(x_1 x_2) \subset \mathbb{A}^2$ as in the picture on the right can be written as the union of the two coordinate axes $X_1 = V(x_2)$ and $X_2 = V(x_1)$, which are themselves affine varieties. However, X_1 and X_2 cannot be decomposed further into finite unions of smaller affine varieties. The following notion generalizes this idea.



$X = X_1 \cup X_2$

Definition 2.6 (Irreducible and connected spaces)**.** Let X be a topological space.

(a) We say that X is **reducible** if it can be written as $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$. Otherwise X is called **irreducible**.

(b) The space X is called **disconnected** if it can be written as $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$ with $X_1 \cap X_2 = \emptyset$. Otherwise X is called **connected**.

Remark 2.7. Although we have given this definition for arbitrary topological spaces, you will usually want to apply the notion of irreducibility only in the Zariski topology. For example, in the classical topology, the complex plane $\mathbb{A}^1_{\mathbb{C}}$ is reducible because it can be written e. g. as the union of closed subsets as

$$\mathbb{A}^1_{\mathbb{C}} = \{z \in \mathbb{C} : |z| \leq 1\} \cup \{z \in \mathbb{C} : |z| \geq 1\}.$$

In the Zariski topology however, $\mathbb{A}^1$ is irreducible by Example 1.6 (as it should be).

In contrast, the notion of connectedness can be used in the "usual" topology too and does mean there what you think it should mean.

In the Zariski topology, the algebraic characterization of the irreducibility and connectedness of affine varieties is the following.

Proposition 2.8. *Let X be a disconnected affine variety, with $X = X_1 \cup X_2$ for two disjoint closed subsets $X_1, X_2 \subsetneq X$. Then $A(X) \cong A(X_1) \times A(X_2)$.*

Proof. By Proposition 1.21 (c) we have $A(X_1) \cong A(X)/I(X_1)$ and $A(X_2) \cong A(X)/I(X_2)$. Hence there is a ring homomorphism

$$\varphi : A(X) \to A(X_1) \times A(X_2), \ \ f \mapsto (\overline{f}, \overline{f}).$$

We have to show that it is bijective.

- φ is injective: If $(\overline{f}, \overline{f}) = (\overline{0}, \overline{0})$ then $f \in I(X_1) \cap I(X_2) = I(X_1 \cup X_2) = I(X) = (0)$ by Lemma 1.26 (b).
- φ is surjective: Let $(\overline{f_1}, \overline{f_2}) \in A(X_1) \times A(X_2)$. Note that
$$A(X) = I(\emptyset) = I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$$
by Lemma 1.26 (a). Thus $1 \in \sqrt{I(X_1) + I(X_2)}$, and hence $1 \in I(X_1) + I(X_2)$, which means $I(X_1) + I(X_2) = A(X)$. We can therefore find $g_1 \in I(X_1)$ and $g_2 \in I(X_2)$ with $f_1 - f_2 = g_1 - g_2$, so that $f_1 - g_1 = f_2 - g_2$. This latter element of $A(X)$ then maps to $(\overline{f_1}, \overline{f_2})$ under φ. □

Proposition 2.9. *An affine variety X is irreducible if and only if $A(X)$ is an integral domain.*

Proof. "$\Rightarrow$": Assume that $A(X)$ is not an integral domain, i. e. there are non-zero $f_1, f_2 \in A(X)$ with $f_1 f_2 = 0$. Then $X_1 = V(f_1)$ and $X_2 = V(f_2)$ are closed, not equal to X (since f_1 and f_2 are non-zero), and $X_1 \cup X_2 = V(f_1) \cup V(f_2) = V(f_1 f_2) = V(0) = X$. Hence X is reducible.

"$\Leftarrow$": Assume that X is reducible, with $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$. By Proposition 1.21 (b) this means that $I(X_i) \neq (0)$ for $i = 1, 2$, and so we can choose non-zero $f_i \in I(X_i)$. Then $f_1 f_2$ vanishes on $X_1 \cup X_2 = X$. Hence $f_1 f_2 = 0 \in A(X)$, i. e. $A(X)$ is not an integral domain. □

Remark 2.10. If X is an affine subvariety of an affine variety Y we know by Proposition 1.21 (c) that $A(X) = A(Y)/I(X)$. So $A(X)$ is an integral domain, i. e. X is irreducible, if and only if for all $f, g \in A(Y)$ the relation $fg \in I(X)$ implies $f \in I(X)$ or $g \in I(X)$. In commutative algebra, ideals with this property are called *prime ideals*. So in other words, in the one-to-one correspondence of Proposition 1.21 (b) between affine subvarieties of Y and radical ideals in $A(Y)$ the irreducible subvarieties correspond exactly to prime ideals.

Example 2.11.

(a) A finite affine variety is irreducible if and only if it is connected: namely if and only if it contains at most one point.

(b) Any irreducible space is connected.

(c) The affine space $\mathbb{A}^n$ is irreducible (and thus connected) by Proposition 2.9 since its coordinate ring $A(\mathbb{A}^n) = K[x_1, \dots, x_n]$ is an integral domain. More generally, this holds for any affine variety given by linear equations, since again its coordinate ring is isomorphic to a polynomial ring.

(d) The union $X = V(x_1 x_2) \subset \mathbb{A}^2$ of the two coordinate axes $X_1 = V(x_2)$ and $X_2 = V(x_1)$ is not irreducible, since $X = X_1 \cup X_2$. But X_1 and X_2 themselves are irreducible by (c). Hence we have decomposed X into a union of two irreducible spaces.

As already announced, we now want to see that such a decomposition into finitely many irreducible spaces is possible for any affine variety. In fact, this works for a much larger class of topological spaces, the so-called *Noetherian* spaces:

Definition 2.12 (Noetherian topological spaces)**.** A topological space X is called **Noetherian** if there is no infinite strictly decreasing chain

$$X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$$

of closed subsets of X.

Lemma 2.13. *Any affine variety is a Noetherian topological space.*

Proof. Let X be an affine variety. Assume that there is an infinite chain $X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$ of subvarieties of X. By Proposition 1.21 (b) there is then a corresponding infinite chain

$$I(X_0) \subsetneq I(X_1) \subsetneq I(X_2) \subsetneq \cdots$$

of ideals in $A(X)$. It is checked immediately that the union $I := \bigcup_{n=0}^{\infty} I(X_i)$ is then an ideal as well [G1, Exercise 10.38 (a)]. By Proposition 1.21 (a) it is finitely generated, i. e. we have $I = (f_1, \dots, f_n)$ for some $f_1, \dots, f_n \in I$. All these polynomials have to lie in one of the ideals $I(X_m)$ — and in fact in the same one since these ideals form a chain. But then we have $I = (f_1, \dots, f_n) \subset I(X_m) \subsetneq I$, a contradiction. □

Remark 2.14 (Subspaces of Noetherian spaces are Noetherian)**.** Let A be a subset of a Noetherian topological space X. Then A is also Noetherian: otherwise we would have an infinite strictly descending chain of closed subsets of A, which by definition of the subspace topology we can write as

$$A \cap Y_0 \supsetneq A \cap Y_1 \supsetneq A \cap Y_2 \supsetneq \cdots$$

for closed subsets $Y_0, Y_1, Y_2, \dots$ of X. Then

$$Y_0 \supset Y_0 \cap Y_1 \supset Y_0 \cap Y_1 \cap Y_2 \supset \cdots$$

is an infinite decreasing chain of closed subsets of X. In fact, in contradiction to our assumption it is also strictly decreasing, since $Y_0 \cap \cdots \cap Y_k = Y_0 \cap \cdots \cap Y_{k+1}$ for some $k \in \mathbb{N}$ would imply $A \cap Y_k = A \cap Y_{k+1}$ by intersecting with A.

Combining Lemma 2.13 with Remark 2.14 we therefore see that any subset of an affine variety is a Noetherian topological space. In fact, all topological spaces occurring in this class will be Noetherian, and thus we can safely restrict our attention to this class of spaces.

Proposition 2.15 (Irreducible decomposition of Noetherian spaces)**.** *Every Noetherian topological space X can be written as a finite union $X = X_1 \cup \cdots \cup X_r$ of irreducible closed subsets. If one assumes that $X_i \not\subset X_j$ for all $i \neq j$, then $X_1, \dots, X_r$ are unique (up to permutation). They are called the* ***irreducible components*** *of X.*

Proof. To prove existence, assume that there is a topological space X for which the statement is false. In particular, X is reducible, hence $X = X_1 \cup X_1'$ as in Definition 2.6 (a). Moreover, the statement of the proposition must be false for at least one of these two subsets, say X_1. Continuing this construction, one arrives at an infinite chain $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$ of closed subsets, which is a contradiction as X is Noetherian.

To show uniqueness, assume that we have two decompositions

$$X = X_1 \cup \cdots \cup X_r = X_1' \cup \cdots \cup X_s'. \qquad (*)$$

Then for any fixed $i \in \{1, \dots, r\}$ we have $X_i \subset \bigcup_j X_j'$, so $X_i = \bigcup_j (X_i \cap X_j')$. But X_i is irreducible, and so we must have $X_i = X_i \cap X_j'$, i. e. $X_i \subset X_j'$ for some j. In the same way we conclude that $X_j' \subset X_k$ for some k, so that $X_i \subset X_j' \subset X_k$. By assumption this is only possible for $i = k$, and consequently $X_i = X_j'$. Hence every set appearing on the left side of $(*)$ also appears on the right side (and vice versa), which means that the two decompositions agree. □

Remark 2.16 (Computation of irreducible decompositions)**.** In general, the actual computation of the irreducible decomposition of an affine variety is quite difficult and requires advanced algorithmic methods of computer algebra. In fact, the corresponding question in commutative algebra is to find the isolated primes of a so-called *primary decomposition* of an ideal [G5, Chapter 8]. But in simple cases the irreducible decomposition might be easy to spot geometrically, as e. g. in Example 2.11 (d).

Exercise 2.17. Find the irreducible components of the affine variety $V(x_1 - x_2x_3, x_1x_3 - x_2^2) \subset \mathbb{A}^3_{\mathbb{C}}$. 03

Remark 2.18 (Open subsets of irreducible spaces are dense)**.** We have already seen in Example 2.5 (a) that open subsets tend to be very "big" in the Zariski topology. Here are two precise statements along these lines. Let X be an irreducible topological space, and let U and U' be non-empty open subsets of X. Then:

(a) The intersection $U \cap U'$ is never empty. In fact, by taking complements this is just equivalent to saying that the union of the two proper closed subsets $X \backslash U$ and $X \backslash U'$ is not equal to X, i. e. to the definition of irreducibility.

(b) The closure $\overline{U}$ of U is all of X — one says that U is *dense* in X. This is easily seen: if $Y \subset X$ is any closed subset containing U then $X = Y \cup (X \backslash U)$, and since X is irreducible and $X \backslash U \neq X$ we must have $Y = X$.

Exercise 2.19. Let A be a subset of a topological space X. Prove:

(a) If $Y \subset A$ is closed in the subspace topology of A then $\overline{Y} \cap A = Y$.

(b) A is irreducible if and only if $\overline{A}$ is irreducible.

Exercise 2.20. Let $\{U_i : i \in I\}$ be an open cover of a topological space X, and assume that $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$. Show:

(a) If U_i is connected for all $i \in I$ then X is connected.

(b) If U_i is irreducible for all $i \in I$ then X is irreducible.

Exercise 2.21. Let $f : X \to Y$ be a continuous map of topological spaces. Prove:

(a) If X is irreducible then so is $f(X)$.

(b) If X is connected then so is $f(X)$.

Exercise 2.22. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be irreducible affine varieties. Prove that the coordinate ring $A(X \times Y)$ of their product is an integral domain, and hence that $X \times Y$ is irreducible as well.

An important application of the notion of irreducibility is the definition of the dimension of an affine variety (or more generally of a topological space — but as with our other concepts above you will only want to apply it to the Zariski topology). Of course, at least in the case of complex varieties we have a geometric idea what the dimension of an affine variety should be: the number of coordinates that you need to describe X locally around any point. Although there are algebraic definitions of dimension that mimic this intuitive one [G5, Proposition 11.31], the standard definition of dimension that we will give here uses only the language of topological spaces. Finally, all these definitions are of course equivalent and describe the intuitive notion of dimension, but it is actually quite hard to prove this rigorously.

The idea to construct the dimension in algebraic geometry using the Zariski topology is rather simple: if X is an *irreducible* topological space, then any closed subset of X not equal to X should have smaller dimension. The resulting definition is the following.

Definition 2.23 (Dimension and codimension)**.** Let X be a non-empty topological space.

(a) The **dimension** $\dim X \in \mathbb{N} \cup \{\infty\}$ is the supremum over all $n \in \mathbb{N}$ such that there is a chain

$$\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subset X$$

of length n of irreducible closed subsets $Y_1, \dots, Y_n$ of X.

(b) If $Y \subset X$ is a non-empty irreducible closed subset of X the **codimension** $\operatorname{codim}_X Y$ of Y in X is again the supremum over all n such that there is a chain

$$Y \subset Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subset X$$

of irreducible closed subsets $Y_1, \dots, Y_n$ of X containing Y.

To avoid confusion, we will always denote the dimension of a K-vector space V by $\dim_K V$, leaving the notation $\dim X$ (without an index) for the dimension of a topological space X as above.

According to the above idea, one should imagine each Y_i as having dimension i in a maximal chain as in Definition 2.23 (a), so that finally $\dim X = n$. In the same way, each Y_i in Definition 2.23 (b) should have dimension $i + \dim Y$ in a maximal chain, so that $n = \dim X - \dim Y$ can be thought of as the difference of the dimensions of X and Y.

Example 2.24.

(a) If X is a (non-empty) finite affine variety then $\dim X = 0$. In fact, since points are closed in X all subsets of X will be closed, and thus the only irreducible closed subsets of X are single points. There are therefore only chains of length 0 of irreducible closed subsets of X.

(b) In contrast to (a), general finite topological spaces need not have dimension 0. For example, the two-pointed topological space $X = \{a,b\}$ whose closed subsets are exactly $\emptyset$, $\{a\}$, and X has dimension 1 since $\{a\} \subsetneq X$ is a chain of length 1 of irreducible closed subsets of X (and there are certainly no longer ones).

However, this will not be of further importance for us since all topological spaces occurring in this class will have the property that points are closed.

(c) By Example 1.6 the affine space $\mathbb{A}^1$ has dimension 1: maximal chains of irreducible closed subsets of $\mathbb{A}^1$ are $\{a\} \subsetneq \mathbb{A}^1$ for any $a \in \mathbb{A}^1$.

(d) It is easy to see that the affine space $\mathbb{A}^n$ for $n \in \mathbb{N}_{>0}$ has dimension *at least* n, since there is certainly a chain

$$V(x_1,\dots,x_n) \subsetneq V(x_2,\dots,x_n) \subsetneq \cdots \subsetneq V(x_n) \subsetneq V(0) = \mathbb{A}^n$$

of irreducible (linear) closed subsets of $\mathbb{A}^n$ of length n.

Of course, we would expect geometrically that the dimension of $\mathbb{A}^n$ is *equal* to n. Although this turns out to be true, the proof of this result is unfortunately rather difficult and technical. It is given in the "Commutative Algebra" class, where dimension is one of the major topics. In fact, our Definition 2.23 is easy to translate into commutative algebra: since irreducible closed subvarieties of an affine variety X correspond exactly to prime ideals in $A(X)$ by Remark 2.10, the dimension of X is the supremum over all n such that there is a chain $I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_n$ of prime ideals in $A(X)$ — and this can be studied algebraically.

Let us now quote the results on the dimension of affine varieties that we will use from commutative algebra. They are all very intuitive: besides the statement that $\dim \mathbb{A}^n = n$ they say that for irreducible affine varieties the codimension of Y in X is in fact the difference of the dimensions of X and Y, and that cutting down an irreducible affine variety by one non-trivial equation reduces the dimension by exactly 1.

Proposition 2.25 (Properties of dimension)**.** *Let X and Y be non-empty irreducible affine varieties.*

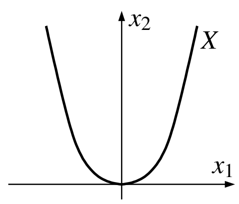
(a) *We have* $\dim(X \times Y) = \dim X + \dim Y$. *In particular,* $\dim \mathbb{A}^n = n$.

(b) *If $Y \subset X$ we have* $\dim X = \dim Y + \operatorname{codim}_X Y$. *In particular,* $\operatorname{codim}_X \{a\} = \dim X$ *for every point $a \in X$.*

(c) *If $f \in A(X)$ is non-zero every irreducible component of $V(f)$ has codimension 1 in X (and hence dimension* $\dim X - 1$ *by (b)).*

Proof. Statement (a) is [G5, Proposition 11.9 (a) and Exercise 11.33 (a)], (b) is [G5, Example 11.13 (a)], and (c) is [G5, Corollary 11.19]. □

Example 2.26. Let $X = V(x_2 - x_1^2) \subset \mathbb{A}^2_{\mathbb{C}}$ be the affine variety whose real points are shown in the picture on the right. Then we have as expected:



(a) X is irreducible by Proposition 2.9 since its coordinate ring $A(X) = \mathbb{C}[x_1,x_2]/(x_2 - x_1^2) \cong \mathbb{C}[x_1]$ is an integral domain.

(b) X has dimension 1 by Proposition 2.25 (c), since it is the zero locus of one non-zero polynomial in the affine space $\mathbb{A}^2$, and $\dim \mathbb{A}^2 = 2$ by Proposition 2.25 (a).

Remark 2.27 (Infinite-dimensional spaces)**.** One might be tempted to think that the "finiteness condition" of a Noetherian topological space X ensures that $\dim X$ is always finite. This is not true however: if we equip the natural numbers $X = \mathbb{N}$ with the topology in which (except $\emptyset$ and X) exactly the subsets $Y_n := \{0, \dots, n\}$ for $n \in \mathbb{N}$ are closed, then X is Noetherian, but has chains $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n$ of non-empty irreducible closed subsets of arbitrary length.

However, Proposition 2.25 (a) together with the following exercise shows that this cannot happen for arbitrary subsets of affine varieties. In fact, all topological spaces considered in this class will have finite dimension.

Exercise 2.28. Let A be an arbitrary subset of a topological space X. Prove that $\dim A \le \dim X$.

Remark 2.29. Depending on where our chains of irreducible closed subvarieties end resp. start, we can break up the supremum in Definition 2.23 into irreducible components or local contributions:

(a) If $X = X_1 \cup \cdots \cup X_r$ is the irreducible decomposition of a Noetherian topological space as in Proposition 2.15, we have

$$\dim X = \max\{\dim X_1, \dots, \dim X_r\}:$$

"$\le$" Assume that $\dim X \ge n$, so that there is a chain $Y_0 \subsetneq \cdots \subsetneq Y_n$ of non-empty irreducible closed subvarieties of X. Then $Y_n = (Y_n \cap X_1) \cup \cdots \cup (Y_n \cap X_r)$ is a union of closed subsets. So as Y_n is irreducible we must have $Y_n = Y_n \cap X_i$, and hence $Y_n \subset X_i$, for some i. But then $Y_0 \subsetneq \cdots \subsetneq Y_n$ is a chain of non-empty irreducible closed subsets in X_i, and hence $\dim X_i \ge n$.

"$\ge$" Let $\max\{\dim X_1, \dots, \dim X_r\} \ge n$. Then there is a chain of non-empty irreducible closed subsets $Y_0 \subsetneq \cdots \subsetneq Y_n$ in some X_i. This is also such a chain in X, and hence $\dim X \ge n$.

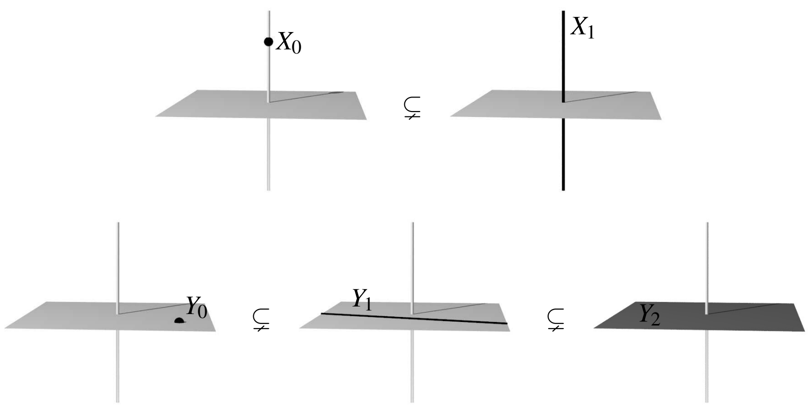
So for many purposes it suffices to consider the dimension of irreducible spaces.

(b) We always have $\dim X = \sup\{\operatorname{codim}_X\{a\} : a \in X\}$:

"$\le$" If $\dim X \ge n$ there is a chain $Y_0 \subsetneq \cdots \subsetneq Y_n$ of non-empty irreducible closed subsets of X. For any $a \in Y_0$ this chain then shows that $\operatorname{codim}_X\{a\} \ge n$.

"$\ge$" If $\operatorname{codim}_X\{a\} \ge n$ for some $a \in X$ there is a chain $\{a\} \subset Y_0 \subsetneq \cdots \subsetneq Y_n$ of non-empty irreducible closed subsets of X, which also shows that $\dim X \ge n$.

The picture below illustrates these two equations: the affine variety $X = V(x_1x_3, x_2x_3) \subset \mathbb{A}^3$ is a union of two irreducible components, a line $V(x_1, x_2)$ of dimension 1 and a plane $V(x_3)$ of dimension 2 (see Proposition 2.25 (a)). So by (a) we have $\dim X = 2$ (with a maximal chain of length 2 as in Definition 2.23 (a) given by $Y_0 \subsetneq Y_1 \subsetneq Y_2$).



As for (b), the codimension of the point Y_0 is 2, whereas the codimension of the point X_0 is 1, as illustrated by the chains in the picture. Note that this codimension of a point can be interpreted geometrically as the *local dimension* of X at this point. Hence Proposition 2.25 (b) can also be interpreted as saying that the local dimension of an irreducible variety is the same at every point.

In practice, we will usually be concerned with affine varieties all of whose components have the same dimension. These spaces have special names that we want to introduce now. Note however that (as with the definition of a variety, see Remark 1.3) these terms are not used consistently throughout the literature — sometimes e. g. a curve is required to be irreducible, and sometimes it might be allowed to have additional components of dimension less than 1.

Definition 2.30 (Pure-dimensional spaces)**.**

(a) A Noetherian topological space X is said to be of **pure dimension** n if every irreducible component of X has dimension n.

(b) An affine variety is called ...

- an **affine curve** if it is of pure dimension 1;
- an **affine surface** if it is of pure dimension 2;
- an **affine hypersurface** of an irreducible affine variety $Y \supset X$ if it is of pure dimension $\dim Y - 1$.

Exercise 2.31. Let X be the set of all 2×3 matrices over a field K that have rank at most 1, considered as a subset of $\mathbb{A}^6 = \mathrm{Mat}(2 \times 3, K)$.

Show that X is an affine variety. Is it irreducible? What is its dimension?

Exercise 2.32. Show that the ideal $I = (x_1x_2, x_1x_3, x_2x_3) \trianglelefteq \mathbb{C}[x_1, x_2, x_3]$ cannot be generated by fewer than three elements. What is the zero locus of I?

Exercise 2.33. Let X be a topological space. Prove:

(a) If $\{U_i : i \in I\}$ is an open cover of X then $\dim X = \sup\{\dim U_i : i \in I\}$.

(b) If X is an irreducible affine variety and $U \subset X$ a non-empty open subset then $\dim X = \dim U$.

Does this statement hold more generally for any irreducible topological space?

Exercise 2.34. Prove the following (maybe at first surprising) statements:

(a) Every affine variety in the real affine space $\mathbb{A}^n_{\mathbb{R}}$ is the zero locus of one polynomial.

(b) Every Noetherian topological space is compact. In particular, every open subset of an affine variety is compact in the Zariski topology. (Recall that by definition a topological space X is compact if every open cover of X has a finite subcover.)

(c) The zero locus of a non-constant polynomial in $\mathbb{C}[x_1, x_2]$ is never compact in the classical topology of $\mathbb{A}^2_{\mathbb{C}} = \mathbb{C}^2$.

(For those of you who know commutative algebra: can you prove that an affine variety over $\mathbb{C}$ containing infinitely many points is *never* compact in the classical topology?)