12. Hilbert Polynomials and Bézout's Theorem

After our study of smooth cubic surfaces in the last chapter, let us now come back to the general theory of algebraic geometry. Our main goal of this chapter will be to determine the number of intersection points of given varieties (in case this number is finite). For example, let X and Y be two plane curves, with (principal) ideals generated by two polynomials f and g, respectively. If they do not have a common irreducible component, their intersection will be zero-dimensional, and we can ask for the number of points in $X \cap Y$. We will see in Bézout's Theorem as in Corollaries 12.20 (b) and 12.26 (b) that this number of points is at most $\deg f \cdot \deg g$, and that we can even make this an equality if we count the points with suitable multiplicities. We have seen a special case of this already in Exercise 4.13, where one of the two curves was a line or a conic.

In particular, this statement means that the number of points in $X \cap Y$ (counted with multiplicities) depends only on the degrees of the defining polynomials, and not on the polynomials themselves. One can view this as a direct generalization of the statement that a degree-d polynomial in one variable always has d zeroes, again counted with multiplicities.

In order to set up a suitable framework for Bézout's Theorem, we have to take note of the following two technical observations:

- As mentioned above, we have to define suitable intersection multiplicities, e. g. for two plane curves X and Y. We have motivated in Remark 1.27 already that such multiplicities are encoded in the (possibly non-radical) ideal $I(X)+I(Y)$. Most constructions in this chapter are therefore based on ideals rather than on varieties, and consequently commutative algebra will play a somewhat greater role than before.
- The simplest example of Bézout's Theorem is that two distinct lines in the plane always meet in one point. This would clearly be false in the affine setting, where two such lines might be parallel. We therefore have to work with projective varieties that can have intersection points at infinity in such cases. 19

Taking these two points into account, we see that our main objects of study will have to be homogeneous ideals in polynomial rings. The central concept that we will need is the Hilbert function of such an ideal.

Definition 12.1 (Hilbert functions)**.**

(a) Let $I \trianglelefteq K[x_0,\dots,x_n]$ be a homogeneous ideal. Then $K[x_0,\dots,x_n]/I$ is a finite-dimensional graded K-algebra by Lemma 6.10 (c). We can therefore define the function

$$h_I : \mathbb{N} \to \mathbb{N},\ d \mapsto \dim_K K[x_0,\dots,x_n]_d/I_d$$

encoding the dimensions of the graded parts of this quotient algebra. It is called the **Hilbert function** of I.

(b) For a projective variety $X \subset \mathbb{P}^n$ we set $h_X := h_{I(X)}$, so that

$$h_X : \mathbb{N} \to \mathbb{N},\ d \mapsto \dim_K S(X)_d,$$

where $S(X) = K[x_0,\dots,x_n]/I(X)$ is the homogeneous coordinate ring of X as in Construction 6.18. We call h_X the Hilbert function of X.

Remark 12.2. Note that the Hilbert function of an ideal is invariant under projective automorphisms as in Example 7.6 (a): an invertible matrix corresponding to an automorphism $\mathbb{P}^n \to \mathbb{P}^n$ also defines an isomorphism $\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$, and hence by Corollary 4.8 an isomorphism $K[x_0,\dots,x_n] \to K[x_0,\dots,x_n]$ of K-algebras. As this isomorphism respects the grading, any ideal has the same Hilbert function as its image under this isomorphism.

Example 12.3.

(a) The Hilbert function of $\mathbb{P}^n$ is given by $h_{\mathbb{P}^n}(d) = \dim_K K[x_0,\dots,x_n]_d = \binom{n+d}{n}$ for $d \in \mathbb{N}$.

(b) Let $I \trianglelefteq K[x_0,\dots,x_n]$ be a homogeneous ideal with $V_p(I) = \emptyset$. Then $\sqrt{I} = (x_0,\dots,x_n)$ or $\sqrt{I} = (1)$ by the projective Nullstellensatz of Proposition 6.22. In both cases we have $x_i^{k_i} \in I$ for suitable $k_i \in \mathbb{N}$ for all i. This means that all monomials of degree at least $k := k_0 + \cdots + k_n$ are contained in I. Hence $I_d = K[x_0,\dots,x_n]_d$ for all $d \geq k$, or in other words

$$h_I(d) = 0 \quad \text{for almost all } d \in \mathbb{N},$$

where as usual we use the term "*almost all*" for "all but finitely many".

(c) Let $X = \{a\} \subset \mathbb{P}^n$ be a single point. To compute its Hilbert function we may assume by Remark 12.2 that this point is $a = (1:0:\cdots:0)$, so that its ideal is $I(a) = (x_1,\dots,x_n)$. Then $S(X) = K[x_0,\dots,x_n]/I(a) \cong K[x_0]$, and hence

$$h_X(d) = 1 \quad \text{for all } d \in \mathbb{N}.$$

Exercise 12.4. Compute the Hilbert function of...

(a) the ideal $(x_0^2x_1^2, x_0^3) \trianglelefteq K[x_0,x_1]$;

(b) two intersecting lines in $\mathbb{P}^3$;

(c) two non-intersecting lines in $\mathbb{P}^3$.

In order to work with Hilbert functions it is convenient to adopt the language of *exact sequences* from commutative algebra. The only statement that we will need about them is the following.

Lemma and Definition 12.5 (Exact sequences)**.** *Let $f : U \to V$ and $g : V \to W$ be linear maps of vector spaces over K. Assume that f is injective, g is surjective, and that $\operatorname{im} f = \ker g$. These assumptions are usually summarized by saying that*

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

is an ***exact sequence*** [G5, Chapter 4].

Then $\dim_K V = \dim_K U + \dim_K W$.

Proof. This is just standard linear algebra: we have

$$\dim_K V = \dim_K \ker g + \dim_K \operatorname{im} g = \dim_K \operatorname{im} f + \dim_K \operatorname{im} g = \dim_K U + \dim_K W,$$

with the last equation following since f is injective and g is surjective. □

Proposition 12.6. *For any two homogeneous ideals $I, J \trianglelefteq K[x_0,\dots,x_n]$ we have*

$$h_{I\cap J} + h_{I+J} = h_I + h_J.$$

Proof. Set $R = K[x_0,\dots,x_n]$. It is easily checked that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/(I\cap J) & \longrightarrow & R/I \times R/J & \longrightarrow & R/(I+J) & \longrightarrow & 0 \\ & & \overline{f} & \longmapsto & (\overline{f},\overline{f}) & & & & \\ & & & & (\overline{f},\overline{g}) & \longmapsto & \overline{f}-\overline{g} & & \end{array}$$

is an exact sequence. Taking its degree-d part and applying Lemma 12.5 gives the desired result. □

Example 12.7.

(a) Let X and Y be disjoint projective varieties in $\mathbb{P}^n$. Then $I(X) \cap I(Y) = I(X \cup Y)$ by Remark 6.23. Moreover, by the same remark the ideal $I(X) + I(Y)$ has empty zero locus since $V(I(X)+I(Y)) = V(I(X)) \cap V(I(Y)) = X \cap Y = \emptyset$, and hence its Hilbert function is almost everywhere zero by Example 12.3 (b). Proposition 12.6 thus implies that

$$h_{X\cup Y}(d) = h_X(d) + h_Y(d) \quad \text{for almost all } d \in \mathbb{N}.$$

In particular, this means by Example 12.3 (c) that for a finite set $X = \{a_1, \dots, a_r\}$ of r points we have

$$h_X(d) = r \quad \text{for almost all } d \in \mathbb{N}.$$

(b) Let $I = (x_1^2) \trianglelefteq K[x_0, x_1]$. It is a non-radical ideal whose projective zero locus consists of the single point $(1:0) \in \mathbb{P}^1$. In fact, it can be viewed as an ideal describing this point "with multiplicity 2" as in Remark 1.27.

The Hilbert function remembers this multiplicity: as $K[x_0, x_1]_d / I_d$ has basis x_0^d and $x_0^{d-1}x_1$ for $d \geq 1$, we see that $h_I(d) = 2$ for almost all d, in the same way as for the Hilbert function of two distinct points as in (a).

(c) Let $X \subset \mathbb{P}^2$ be the union of three points lying on a line. Then there is a homogeneous linear polynomial in $K[x_0, x_1, x_2]$ vanishing on X, so that $\dim_K I(X)_1 = 1$. Hence $h_X(1) = \dim_K K[x_0, x_1, x_2]_1 / I(X)_1 = 3 - 1 = 2$. On the other hand, if X consists of three points not on a line, then no linear polynomial vanishes on X, and consequently $h_X(1) = \dim_K K[x_0, x_1, x_2]_1 / I(X)_1 = 3 - 0 = 3$. So in particular, we see that in contrast to Remark 12.2 the Hilbert function is not invariant under arbitrary isomorphisms, since any set of three points is isomorphic to any other such set.

Together with the result of (a), for a finite set $X \subset \mathbb{P}^n$ we can say intuitively that $h_X(d)$ encodes the *number* of points in X for large values of d, whereas it gives some information on the relative *position* of these points for small values of d.

Note that the intersection $X \cap Y$ of two varieties X and Y corresponds to the sum of their ideals. To obtain a formula for the number of points in $X \cap Y$ we therefore have to compute the Hilbert functions of sums of ideals. The following lemma will help us to do this in the case when one of the ideals is principal.

Lemma 12.8. *Let $I \trianglelefteq K[x_0, \dots, x_n]$ be a homogeneous ideal, and let $f \in K[x_0, \dots, x_n]$ be a homogeneous polynomial of degree e. Assume that there is a number $d_0 \in \mathbb{N}$ with the following property:*

for all homogeneous $g \in K[x_0, \dots, x_n]$ of degree at least d_0 with $fg \in I$ we have $g \in I$.

Then $h_{I+(f)}(d) = h_I(d) - h_I(d-e)$ for almost all $d \in \mathbb{N}$.

Proof. Let $R = K[x_0, \dots, x_n]$. There is an exact sequence

$$0 \longrightarrow R_{d-e}/I_{d-e} \xrightarrow{\cdot f} R_d/I_d \longrightarrow R_d/(I+(f))_d \longrightarrow 0$$

for all d with $d - e \geq d_0$, where the second map is just the quotient map. In fact, it is obvious that this quotient map is surjective, and that its kernel is exactly the image of the first map. The injectivity of the first map is precisely the assumption of the lemma.

The desired statement now follows immediately from Lemma 12.5. □

Before we can apply this lemma, we have to analyze the geometric meaning of the somewhat technical assumption that $fg \in I$ implies $g \in I$ for all polynomials g (of sufficiently large degree).

Remark 12.9. Assume that $I = I(X)$ is the (radical) ideal of a projective variety X. Consider the irreducible decomposition $X = X_1 \cup \dots \cup X_r$ of X, so that $I = I(X_1) \cap \dots \cap I(X_r)$ by Remark 6.23 (c).

We claim that the assumption of Lemma 12.8 is then satisfied if f does not vanish identically on any X_i. In fact, in this case f is non-zero in the integral domain $S(X_i)$ for all i (see Exercise 6.31 (b)). Hence $gf \in I$, i. e. $gf = 0 \in S(X_i)$, implies $g = 0 \in S(X_i)$ for all i, and thus $g \in I$.

If I is not radical, a similar statement holds — but in order for this to work we need to be able to decompose I as an intersection of ideals corresponding to irreducible varieties again. This so-called primary decomposition of I is one of the main topics in the Commutative Algebra class [G5, Chapter 8]. We will therefore just quote the results that we are going to need.

Remark 12.10 (*Primary decompositions*)**.** Let $I \trianglelefteq K[x_0,\dots,x_n]$ be an arbitrary ideal. Then there is a decomposition

$$I = I_1 \cap \cdots \cap I_r$$

into *primary ideals* $I_1,\dots,I_r$, which means by definition that $gf \in I_i$ implies $g \in I_i$ or $f \in \sqrt{I_i}$ for all i and all polynomials $f,g \in K[x_0,\dots,x_n]$ [G5, Definition 8.9 and Proposition 8.16]. Moreover, this decomposition satisfies the following properties:

(a) The zero loci $V_a(I_i)$ are irreducible: by Proposition 2.9 and the Nullstellensatz, this is the same as saying that $K[x_0,\dots,x_n]/\sqrt{I_i}$ is an integral domain. So assume that f and g are two polynomials with $gf \in \sqrt{I_i}$. Then $g^k f^k \in I_i$ for some $k \in \mathbb{N}$. But this implies that $g^k \in I_i$ or $f^k \in \sqrt{I_i}$ since I_i is primary, and hence that $g \in \sqrt{I_i}$ or $f \in \sqrt{\sqrt{I_i}} = \sqrt{I_i}$.

(b) Applying Remark 1.25 (c) to our decomposition, we see that

$$V_a(I) = V_a(I_1) \cup \cdots \cup V_a(I_r).$$

In particular, by (a) all irreducible components of $V_a(I)$ must be among the varieties $V_a(I_1),\dots,V_a(I_r)$. Moreover, we can assume that no two of these varieties coincide: if $V_a(I_i) = V_a(I_j)$ for some $i \neq j$, i.e. by the Nullstellensatz $\sqrt{I_i} = \sqrt{I_j}$, we can replace the two ideals I_i and I_j by the single ideal $I_i \cap I_j$ in the decomposition, which is easily seen to be primary again.

However, it may well happen that there are (irreducible) varieties among $V_a(I_1),\dots,V_a(I_r)$ that are strictly contained in an irreducible component of $V_a(I)$ [G5, Example 8.23]. These varieties are usually called the *embedded components* of I. In the primary decomposition, the ideals corresponding to the irreducible components are uniquely determined, whereas the ones corresponding to the embedded components are usually not [G5, Example 8.23 and Proposition 8.34].

Using these primary decompositions, we can now show for a homogeneous ideal $I \trianglelefteq K[x_0,\dots,x_n]$ that, by a suitable homogeneous linear change of coordinates, we can always achieve that $f = x_0$ satisfies the condition of Lemma 12.8. In fact, assume that g is a homogeneous polynomial such that $gx_0 \in I_i$ for all i. We distinguish two cases:

- If $V_a(I_i) \subset \{0\}$ then $\sqrt{I_i} \supset (x_0,\dots,x_n)$ by the Nullstellensatz. Hence $K[x_0,\dots,x_n]_d \subset I_i$ for large d in the same way as in Example 12.3 (b), which means that $g \in I_i$ if the degree of g is big enough.
- If $V_a(I_i) \not\subset \{0\}$ a general homogeneous linear change of coordinates will assure that $V_a(I_i)$ is not contained in the hypersurface $V_a(x_0)$. Then x_0 is not identically zero on $V_a(I_i)$, so that $x_0 \notin I_a(V_a(I_i)) = \sqrt{I_i}$. Since I_i is primary, we conclude that $g \in I_i$.

Let us now come back to our study of Hilbert functions. We have already seen that the important information in h_I concerning the number of intersection points of varieties is contained in its values $h_I(d)$ for large d. We therefore have to study the behavior of $h_I(d)$ as $d \to \infty$. The central result in this direction is that the Hilbert function is eventually polynomial, with particularly the degree and the leading coefficient of this polynomial deserving special attention.

Proposition and Definition 12.11 (Hilbert polynomials)**.** *Let $I \trianglelefteq K[x_0,\dots,x_n]$ be a homogeneous ideal. Then there is a unique polynomial $\chi_I \in \mathbb{Q}[d]$ such that $\chi_I(d) = h_I(d)$ for almost all $d \in \mathbb{N}$. Moreover,*

(a) *The degree of χ_I is $m := \dim V_p(I)$.*

(b) *If $V_p(I) \neq \emptyset$, the leading coefficient of χ_I is $\frac{1}{m!}$ times a positive integer.*

*The polynomial χ_I is called the **Hilbert polynomial** of I. For a projective variety $X \subset \mathbb{P}^n$ we set $\chi_X := \chi_{I(X)}$.*

Proof. It is obvious that a polynomial with infinitely many fixed values is unique. So let us prove the existence of χ_I by induction on $m = \dim V_p(I)$. The start of the induction follows from Example 12.3 (b): for $V_p(I) = \emptyset$ we obtain the zero polynomial χ_I.

Let us now assume that $V_p(I) \neq \emptyset$. By a homogeneous linear change of coordinates (which does not affect the Hilbert function by Remark 12.2) we can assume that the polynomial x_0 does not vanish identically on any irreducible component of $V_p(I)$. Hence $\dim V_p(I + (x_0)) \leq m-1$, and so by our induction on m we know that $d \mapsto h_{I+(x_0)}(d)$ is a polynomial of degree at most $m-1$ for large d. For reasons that will be apparent later, let us choose $\binom{d}{0}, \dots, \binom{d}{m-1}$ as a basis of the vector space of polynomials in d of degree at most $m-1$, so that for suitable $c_0, \dots, c_{m-1} \in \mathbb{Q}$ we can write

$$h_{I+(x_0)}(d) = \sum_{i=0}^{m-1} c_i \binom{d}{i} \quad \text{for almost all } d \in \mathbb{N}.$$

Moreover, by Remark 12.10 we can assume that Lemma 12.8 is applicable for $f = x_0$, so that

$$h_I(d) - h_I(d-1) = h_{I+(x_0)}(d) = \sum_{i=0}^{m-1} c_i \binom{d}{i} \quad \text{for almost all } d \in \mathbb{N}. \tag{1}$$

We will now show by induction on d that there is a constant $c \in \mathbb{Q}$ such that

$$h_I(d) = c + \sum_{i=0}^{m-1} c_i \binom{d+1}{i+1} \quad \text{for almost all } d \in \mathbb{N}. \tag{2}$$

The start of the induction is trivial, since we can always adjust c so that this equation holds at a single value of d (chosen so that (1) holds for all larger values of d). But then for all larger d we have

$$h_I(d+1) = h_I(d) + \sum_{i=0}^{m-1} c_i \binom{d+1}{i} = c + \sum_{i=0}^{m-1} c_i \left(\binom{d+1}{i+1} + \binom{d+1}{i} \right) = c + \sum_{i=0}^{m-1} c_i \binom{d+2}{i+1}$$

by (1) and the induction assumption. As the right hand side of (2) is a polynomial in d (of degree at most m), this proves the existence of χ_I.

Finally, if $V_p(I) \neq \emptyset$ let us show that the d^m-coefficient of χ_I is $\frac{1}{m!}$ times a positive integer, thus proving the additional statements (a) and (b). We will do this again by induction on m.

- $m = 0$: In this case χ_I is a constant, and it is clearly a non-negative integer, since it is by definition the dimension of $K[x_0, \dots, x_n]_d / I_d$ for large d. Moreover, it cannot be zero, since otherwise $I_d = K[x_0, \dots, x_n]_d$ for some d, which implies $x_i^d \in I_d$ for all i and thus $V_p(I) = \emptyset$.
- $m > 0$: In this case $V_p(I)$ has an irreducible component of dimension m (and none of bigger dimension). In our proof above, the zero locus of x_0 on this component is non-empty by Exercise 6.32 (b), and of dimension $m-1$ by Proposition 2.25 (c). Hence $\dim V_p(I + (x_0)) = m-1$, and so by induction $\chi_{I+(x_0)}$ is a polynomial of degree $m-1$, with $(m-1)!$ times the leading coefficient being a positive integer. But note that in the proof above this integer is just c_{m-1}, which is then also $m!$ times the leading coefficient of χ_I by (2). □

Remark 12.12. Of course, all our statements concerning the values of the Hilbert function $d \mapsto h_I(d)$ at large values of d can be transferred immediately to the Hilbert polynomial. For example, Example 12.3 implies that $\chi_I = 0$ (as a polynomial) if $V_p(I) = \emptyset$, and $\chi_I = 1$ if I is the ideal of a point. Similar statements hold for Proposition 12.6, Example 12.7, and Lemma 12.8.

Definition 12.13 (Degree)**.** Let $I \trianglelefteq K[x_0, \dots, x_n]$ be a homogeneous ideal with non-empty projective zero locus, and let $m = \dim V_p(I)$. Then $m!$ times the leading coefficient of χ_I, which is a positive integer by Proposition 12.11, is called the **degree** $\deg I$ of I. The reason for this name will become clear in Example 12.17.

For a projective variety X, its degree is defined as $\deg X := \deg I(X)$.

Example 12.14.

(a) The degree of $\mathbb{P}^n$ is $n!$ times the d^n-coefficient of $\dim_K K[x_0, \dots, x_n]_d = \binom{n+d}{n}$, i. e. $\deg \mathbb{P}^n = 1$. By Example 12.3 (c), the degree of a single point is 1 as well.

More generally, if $X \subset \mathbb{P}^n$ is any linear space then its homogeneous coordinate ring is isomorphic to a polynomial ring, and hence as above $\deg X = 1$ again.

(b) Let X and Y be projective varieties in $\mathbb{P}^n$ of the same dimension m, and assume that they do not have a common irreducible component. Then the zero locus $X \cap Y$ of $I(X) + I(Y)$ has dimension smaller than m, so that $\chi_{I(X)+I(Y)}$ has degree less than m. Moreover, we have $I(X) \cap I(Y) = I(X \cup Y)$, and hence considering $m!$ times the degree-m coefficients in the Hilbert polynomials of $I(X) \cap I(Y)$, $I(X) + I(Y)$, $I(X)$, and $I(Y)$ yields

$$\deg(X \cup Y) = \deg X + \deg Y$$

by Proposition 12.6 (which of course holds for the Hilbert polynomials as well as for the Hilbert functions).

(c) Let $I \trianglelefteq K[x_0, \dots, x_n]$ be a homogeneous ideal with finite zero locus, consisting of r points. Then $\chi_{\sqrt{I}} = r$ by Example 12.7 (a). But $\sqrt{I} \supset I$ also implies $\chi_{\sqrt{I}} \le \chi_I$, and so we conclude that

$$\deg I = \chi_I \ge \chi_{\sqrt{I}} = r.$$

In fact, in Corollary 12.26 we will refine this statement by interpreting $\deg I$ as a sum of multiplicities for each point in $V_p(I)$, with each of these multiplicities being a positive integer.

Exercise 12.15.

(a) Show that the degree of the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^n$ is $\binom{n+m}{n}$.

(b) Show that the degree of the degree-d Veronese embedding of $\mathbb{P}^n$ is d^n.

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We are now ready to prove the main result of this chapter.

Proposition 12.16 (Bézout's Theorem). *Let $X \subset \mathbb{P}^n$ be a projective variety of dimension at least 1, and let $f \in K[x_0, \dots, x_n]$ be a homogeneous polynomial that does not vanish identically on any irreducible component of X. Then*

$$\deg(I(X) + (f)) = \deg X \cdot \deg f.$$

Proof. Let $m = \dim X$. By Definition 12.13, the Hilbert polynomial of X is given by

$$\chi_X(d) = \frac{\deg X}{m!} d^m + a\, d^{m-1} + (\text{terms of degree less than } m-1)$$

for some $a \in \mathbb{Q}$. So by Remark 12.9 we can apply Lemma 12.8 and obtain with $e := \deg f$

$$\begin{aligned}
\chi_{I(X)+(f)}(d) &= \chi_X(d) - \chi_X(d-e) \\
&= \frac{\deg X}{m!} (d^m - (d-e)^m) + a\,(d^{m-1} - (d-e)^{m-1}) + (\text{terms of degree less than } m-1) \\
&= \frac{e \deg X}{(m-1)!} d^{m-1} + (\text{terms of degree less than } m-1).
\end{aligned}$$

By Definition 12.13 again, this means that $\deg(I(X) + (f)) = e \deg X = \deg X \cdot \deg f$. □

Example 12.17. Let $I = (f) \trianglelefteq K[x_0, \dots, x_n]$ be a principal ideal. Then Bézout's Theorem together with Example 12.14 (a) implies

$$\deg I = \deg((0) + (f)) = \deg \mathbb{P}^n \cdot \deg f = \deg f.$$

In particular, if $X \subset \mathbb{P}^n$ is a hypersurface, so that $I(X) = (f)$ for some homogeneous polynomial f by Exercise 7.16 (a), then $\deg X = \deg f$. This justifies the name "degree" in Definition 12.13.

Exercise 12.18. Prove that every pure-dimensional projective variety of degree 1 is a linear space.

Notation 12.19. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. In certain cases there are commonly used names to describe the degree and / or the dimension of X that we have probably used informally already several times:

(a) If $\deg X = 1$ then X is called a **line** if $\dim X = 1$, a **plane** if $\dim X = 2$, and a **hyperplane** if $\dim X = n-1$.

(b) For any dimension, X is called a **quadric** if $\deg X = 2$, a **cubic** if $\deg X = 3$, a **quartic** if $\deg X = 4$, and so on.

Corollary 12.20 (Bézout's Theorem for curves)**.**

(a) *Let $X \subset \mathbb{P}^n$ be a curve, and let $f \in K[x_0,\dots,x_n]$ be a homogeneous polynomial that does not vanish identically on any irreducible component of X. Then*

$$|X \cap V(f)| \le \deg X \cdot \deg f.$$

(b) *For any two curves X and Y in $\mathbb{P}^2$ without a common irreducible component we have*

$$|X \cap Y| \le \deg X \cdot \deg Y.$$

Proof.

(a) As $I(X)+(f)$ is an ideal with zero locus $X \cap V(f)$, the statement follows from Bézout's Theorem together with Example 12.14 (c).

(b) Apply (a) to a polynomial f generating $I(Y)$, and use Example 12.17. □

For the remaining part of this chapter we will focus on the case of curves as in Corollary 12.20. Our goal is to assign a natural multiplicity to each point in $X \cap V(f)$ (resp. $X \cap Y$) so that the inequality becomes an equality when all points are counted with their respective multiplicities. In order to achieve this we have to study the degree of a homogeneous ideal with zero-dimensional zero locus from a local point of view. It is convenient to do this in an affine chart of $\mathbb{P}^n$, and then finally in the local rings.

Exercise 12.21. Let $I \trianglelefteq K[x_0,\dots,x_n]$ be a homogeneous ideal with $\dim V_p(I) = 0$. Assume that we have chosen coordinates so that all points in $V_p(I)$ have a non-vanishing x_0-coordinate. Prove that the degree of I is then

$$\deg I = \chi_I = \dim_K K[x_1,\dots,x_n]/J,$$

where $J = \{f(1,x_1,\dots,x_n) : f \in I\} \trianglelefteq K[x_1,\dots,x_n]$.

The following lemma now expresses this dimension as a sum of local dimensions. In case you have attended the Commutative Algebra class already you will probably recognize this as precisely the Structure Theorem for Artinian rings, stating that an Artinian ring is always the product of its localizations [G5, Proposition 7.20].

Lemma 12.22. *Let $J \trianglelefteq K[x_1,\dots,x_n]$ be an ideal with finite affine zero locus $V_a(J) = \{a_1,\dots,a_r\}$. Then*

$$K[x_1,\dots,x_n]/J \cong \mathscr{O}_{\mathbb{A}^n,a_1}/J\,\mathscr{O}_{\mathbb{A}^n,a_1} \times \cdots \times \mathscr{O}_{\mathbb{A}^n,a_r}/J\,\mathscr{O}_{\mathbb{A}^n,a_r},$$

where $J\,\mathscr{O}_{\mathbb{A}^n,a_i}$ denotes the ideal in $\mathscr{O}_{\mathbb{A}^n,a_i}$ generated by all elements $\frac{f}{1}$ for $f \in J$.

Proof. Consider the primary decomposition of J as in Remark 12.10. By part (b) of this remark it is of the form $J = J_1 \cap \cdots \cap J_r$ for some ideals $J_1,\dots,J_r$ with $V_a(J_i) = \{a_i\}$ for all i. Moreover, note that $J_i\,\mathscr{O}_{\mathbb{A}^n,a_j}$ is the unit ideal for $i \ne j$ since $a_j \notin V_a(J_i)$ implies that there is a polynomial in J_i not vanishing at a_j, so that it is a unit in the local ring $\mathscr{O}_{\mathbb{A}^n,a_j}$. Hence it suffices to prove that

$$K[x_1,\dots,x_n]/J \cong \mathscr{O}_{\mathbb{A}^n,a_1}/J_1\,\mathscr{O}_{\mathbb{A}^n,a_1} \times \cdots \times \mathscr{O}_{\mathbb{A}^n,a_r}/J_r\,\mathscr{O}_{\mathbb{A}^n,a_r}.$$

We will do this by showing that the K-algebra homomorphism

$$\varphi : K[x_1,\dots,x_n]/J \to \mathscr{O}_{\mathbb{A}^n,a_1}/J_1\,\mathscr{O}_{\mathbb{A}^n,a_1} \times \cdots \times \mathscr{O}_{\mathbb{A}^n,a_r}/J_r\,\mathscr{O}_{\mathbb{A}^n,a_r},\ \ \overline{f} \mapsto \left(\overline{f},\dots,\overline{f}\right)$$

is bijective.

- φ is injective: Let f be a polynomial with $\varphi(\overline{f}) = 0$. Then f lies in $J_i \mathscr{O}_{\mathbb{A}^n,a_i}$ for all i, i. e. $\frac{f}{1} = \frac{g_i}{f_i}$ for some g_i, f_i with $g_i \in J_i$ and $f_i \in K[x_0,\dots,x_n]$ such that $f(a_i) \neq 0$. This means that $h_i(f_i f - g_i) = 0$ for some h_i with $h_i(a_i) \neq 0$, and hence $h_i f_i f \in J_i$. But $h_i f_i \notin I(a_i)$ means $h_i f_i \notin \sqrt{J_i}$, and thus $f \in J_i$ since J_i is primary. As this holds for all i, we conclude that $f \in J$, i. e. $\overline{f} = 0$ in $K[x_1,\dots,x_n]/J$.
- φ is surjective: By symmetry of the factors it suffices to prove that $(1,0,\dots,0) \in \operatorname{im}\varphi$. As $V(J_1 + J_i) = \{a_1\} \cap \{a_i\} = \emptyset$ for all $i > 1$ we see that $1 \in \sqrt{J_1 + J_i}$, and hence also $1 \in J_1 + J_i$. There are thus $a_i \in J_1$ and $b_i \in J_i$ with $a_i + b_i = 1$, so that $b_i \equiv 0 \mod J_i$ and $b_i \equiv 1 \mod J_1$. Hence the product $b_2 \cdot \dots \cdot b_r$ is an inverse image of $(1,0,\dots,0)$ under φ. □

It is now straightforward to translate Bézout's Theorem for curves into a local version.

Definition 12.23 (Multiplicities)**.**

(a) Let $I \trianglelefteq K[x_0,\dots,x_n]$ be a homogeneous ideal with finite projective zero locus, and let $a \in \mathbb{P}^n$. Choose an affine patch of $\mathbb{P}^n$ containing a, and let J be the corresponding affine ideal as in Exercise 12.21. Then

$$\operatorname{mult}_a(I) := \dim_K \mathscr{O}_{\mathbb{A}^n,a}/J\,\mathscr{O}_{\mathbb{A}^n,a}$$

is called the **multiplicity** of I at a.

(b) Let $X \subset \mathbb{P}^n$ be a projective curve, and let $a \in X$ be a point. For a homogeneous polynomial $f \in K[x_0,\dots,x_n]$ that does not vanish identically on any irreducible component of X, the number

$$\operatorname{mult}_a(X,f) := \operatorname{mult}_a(I(X) + (f))$$

is called the **(vanishing) multiplicity** of f at a. Note that $\operatorname{mult}_a(X,f)$ depends only on the class of f modulo $I(X)$ and not on f itself, so that we can also construct the multiplicity $\operatorname{mult}_a(X,f)$ for $f \in S(X)$. In this case, we will also often simplify its notation to $\operatorname{mult}_a(f)$.

If $n = 2$ and $Y \subset \mathbb{P}^2$ is another curve that does not share a common irreducible component with X, the **intersection multiplicity** of X and Y at a is defined as

$$\operatorname{mult}_a(X,Y) := \operatorname{mult}_a(I(X) + I(Y)).$$

Remark 12.24 (Positivity of multiplicities)**.** Continuing the notation of Definition 12.23 (a), note that $1 \notin J\,\mathscr{O}_{\mathbb{A}^n,a}$ if and only if $a \in V_p(I)$. It follows that $\operatorname{mult}_a(I) \geq 1$ if and only if $a \in V_p(I)$. Applying this to Definition 12.23 (b), we see that the vanishing multiplicity $\operatorname{mult}_a(X,f)$ is at least 1 if and only if $f(a) = 0$, and the intersection multiplicity $\operatorname{mult}_a(X,Y)$ is at least 1 if and only if $a \in X \cap Y$. In fact, we will show in Exercise 12.27 that there is also an easy geometric criterion for when $\operatorname{mult}_a(X,Y) = 1$.

Remark 12.25 (Vanishing and intersection multiplicities in local rings)**.** It is often useful to express the multiplicities of Definition 12.23 (b) in terms of local rings as in Definition 12.23 (a). As above, we choose an affine patch $\{x \in \mathbb{P}^n : x_i \neq 0\} \cong \mathbb{A}^n$ of $\mathbb{P}^n$ containing a. By abuse of notation, if $f \in K[x_0,\dots,x_n]$ is a homogeneous polynomial, we will also denote by f the (not necessarily homogeneous) polynomial obtained from it by setting x_i equal to 1, and then also its quotient by 1 in the local ring $\mathscr{O}_{\mathbb{A}^n,a}$ (see Exercise 3.24). Then Definition 12.23 can be formulated as follows:

(a) Let $X \subset \mathbb{P}^n$ be a curve, and let $f \in K[x_0,\dots,x_n]$ be a homogeneous polynomial not vanishing identically on any irreducible component of X. Denote by $U = X \cap \mathbb{A}^n$ the affine part of X in the chosen patch, and let $J = I(U)$ be its ideal. Then the vanishing multiplicity of f at $a \in \mathbb{P}^n$ is equal to $\dim_K \mathscr{O}_{\mathbb{A}^n,a}/(J+(f))\,\mathscr{O}_{\mathbb{A}^n,a}$ by Definition 12.23. But $\mathscr{O}_{\mathbb{A}^n,a}/J\,\mathscr{O}_{\mathbb{A}^n,a} \cong \mathscr{O}_{X,a}$ by Exercise 3.23, and so we conclude that

$$\operatorname{mult}_a(X,f) = \dim_K \mathscr{O}_{X,a}/(f).$$

Note that we could use the same formula to define the vanishing multiplicity for any local function $f \in \mathscr{O}_{X,a}$ that does not vanish identically on any irreducible component of X through a. In fact, for an irreducible variety we will even define such a multiplicity for rational functions in Construction 14.5, which then includes the case of local functions (see Exercise 9.8 (b) and Remark 14.7).

(b) For two curves $X,Y \subset \mathbb{P}^2$ without common irreducible component and ideals $I(X) = (f)$ and $I(X) = (g)$ their intersection multiplicity is

$$\operatorname{mult}_a(X,Y) = \dim_K \mathscr{O}_{\mathbb{A}^2,a}/(f,g),$$

or alternatively with (a)

$$\operatorname{mult}_a(X,Y) = \dim_K \mathscr{O}_{X,a}/(g) = \dim_K \mathscr{O}_{Y,a}/(f).$$

Of course, as we have defined the multiplicities above using affine charts, we could construct them equally well for affine instead of projective varieties. However, the projective case is needed for the local version of Bézout's Theorem, which we can now prove.

Corollary 12.26 (Bézout's Theorem for curves, local version)**.**

(a) *Let $X \subset \mathbb{P}^n$ be a curve, and let $f \in K[x_0,\dots,x_n]$ be a homogeneous polynomial that does not vanish identically on any irreducible component of X. Then*

$$\sum_{a \in X \cap V(f)} \operatorname{mult}_a(X,f) = \deg X \cdot \deg f.$$

(b) *For any two curves X and Y in $\mathbb{P}^2$ without a common irreducible component we have*

$$\sum_{a \in X \cap Y} \operatorname{mult}_a(X,Y) = \deg X \cdot \deg Y.$$

Proof.

(a) By Exercise 12.21, Lemma 12.22, and the definition of multiplicities, all applied to the ideal $I(X) + (f)$, we have

$$\deg(I(X)+(f)) = \sum_{a \in X \cap V(f)} \operatorname{mult}_a(X,f).$$

Hence the statement follows immediately from Proposition 12.16.

(b) This follows from (a) for f a polynomial generating $I(Y)$, since $\deg Y = \deg f$ by Example 12.17. □

Exercise 12.27 (Geometric interpretation of intersection multiplicities)**.** Let $X,Y \subset \mathbb{A}^2$ be two affine curves containing the origin. Moreover, let $I(X) = (f)$ and $I(Y) = (g)$ be their ideals. Show that the following statements are equivalent:

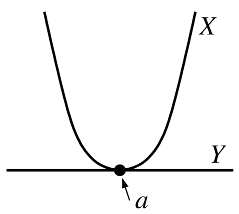
(a) $\dim_K \mathscr{O}_{\mathbb{A}^2,0}/(f,g) = 1$ (i. e. the intersection multiplicity of X and Y at the origin is 1).

(b) X and Y are smooth at 0 and have different tangent spaces there (i. e. "X and Y intersect transversely at the origin").

Example 12.28. Consider the two projective curves

$$X = V(x_0x_2 - x_1^2) \quad \text{and} \quad Y = V(x_2)$$

in $\mathbb{P}^2$, whose affine parts (which we have already considered in Remark 1.27) are shown in the picture on the right. Note that $\deg X = 2$ and $\deg Y = 1$ by Example 12.17, and that $a := (1:0:0)$ is the only point in the intersection $X \cap Y$. As X and Y have the same tangent space at a, we must have $\operatorname{mult}_a(X,Y) \geq 2$ by Exercise 12.27.



In fact, it is easy to compute $\operatorname{mult}_a(X,Y)$ explicitly: by definition we have

$$\operatorname{mult}_a(X,Y) = \operatorname{mult}_a(x_0x_2 - x_1^2, x_2) = \dim_K \mathscr{O}_{\mathbb{A}^2,0}/(x_2 - x_1^2, x_2) = \dim_K \mathscr{O}_{\mathbb{A}^2,0}/(x_1^2, x_2),$$

and since a is the only intersection point of X and Y we can rewrite this by Lemma 12.22 as

$$\operatorname{mult}_a(X,Y) = \dim_K K[x_1,x_2]/(x_1^2,x_2) = \dim_K K[x_1]/(x_1^2) = 2.$$

Note that this is in accordance with Bézout's Theorem as in Corollary 12.26 (b), since $\operatorname{mult}_a(X,Y) = 2 = \deg X \cdot \deg Y$.