1. Affine Varieties

As explained in the introduction, the goal of algebraic geometry is to study solutions of polynomial equations in several variables over a fixed ground field. So let us now make the corresponding definitions.

Convention 1.1. Throughout these notes, K will always denote a fixed base field (which we will require to be algebraically closed after our discussion of Hilbert's Nullstellensatz in Proposition 1.17). Rings are always assumed to be commutative with a multiplicative unit 1. By $K[x_1,\dots,x_n]$ we will denote the *polynomial ring* in n variables $x_1,\dots,x_n$ over K, i. e. the ring of finite formal sums

$$\sum_{i_1,\dots,i_n\in\mathbb{N}} a_{i_1,\dots,i_n}\, x_1^{i_1}\cdot\,\cdots\,\cdot x_n^{i_n}$$

with all $a_{i_1,\dots,i_n}\in K$ (see e. g. [G1, Chapter 9] how this concept of "formal sums" can be defined in a mathematically rigorous way).

Definition 1.2 (Affine varieties)**.**

(a) We call

$$\mathbb{A}^n := \mathbb{A}^n_K := \{(c_1,\dots,c_n) : c_i\in K \text{ for } i=1,\dots,n\}$$

the **affine n-space** over K.

Note that $\mathbb{A}^n_K$ is just K^n as a set. It is customary to use two different notations here since K^n is also a K-vector space and a ring. We will usually use the notation $\mathbb{A}^n_K$ if we want to ignore these additional structures: for example, addition and scalar multiplication are defined on K^n, but not on $\mathbb{A}^n_K$. The affine space $\mathbb{A}^n_K$ will be the ambient space for our zero loci of polynomials below.

(b) For a polynomial

$$f=\sum_{i_1,\dots,i_n\in\mathbb{N}} a_{i_1,\dots,i_n}\, x_1^{i_1}\cdot\,\cdots\,\cdot x_n^{i_n}\quad\in K[x_1,\dots,x_n]$$

and a point $c=(c_1,\dots,c_n)\in\mathbb{A}^n$ we define the **value** of f at c to be

$$f(c)=\sum_{i_1,\dots,i_n\in\mathbb{N}} a_{i_1,\dots,i_n}\, c_1^{i_1}\cdot\,\cdots\,\cdot c_n^{i_n}\quad\in K.$$

If there is no risk of confusion we will often denote a point in $\mathbb{A}^n$ by the same letter x as we used for the formal variables, writing $f\in K[x_1,\dots,x_n]$ for the polynomial and $f(x)$ for its value at a point $x\in\mathbb{A}^n_K$.

(c) For a subset $S\subset K[x_1,\dots,x_n]$ of polynomials we call

$$V(S) := \{x\in\mathbb{A}^n : f(x)=0 \text{ for all } f\in S\}\quad\subset\mathbb{A}^n$$

the (affine) **zero locus** of S. Subsets of $\mathbb{A}^n$ of this form are called **(affine) varieties**. If $S=\{f_1,\dots,f_k\}$ is a finite set, we will write $V(S)=V(\{f_1,\dots,f_k\})$ also as $V(f_1,\dots,f_k)$.

Remark 1.3. Some authors refer to zero loci of polynomials in $\mathbb{A}^n$ as in Definition 1.2 (c) as *(affine) algebraic sets*, reserving the name "affine variety" for such zero loci that are in addition *irreducible* (a concept that we will introduce in Definition 2.6 (a)).

Example 1.4. Here are some simple examples of affine varieties:

(a) Affine n-space itself is an affine variety, since $\mathbb{A}^n=V(0)$. Similarly, the empty set $\emptyset=V(1)$ is an affine variety.

(b) Any single point in $\mathbb{A}^n$ is an affine variety: we have $(c_1,\dots,c_n)=V(x_1-c_1,\dots,x_n-c_n)$.

(c) Linear subspaces of $\mathbb{A}^n = K^n$ are affine varieties.

(d) If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine varieties then so is the *product* $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$.

(e) All examples from the introduction in Chapter 0 are affine varieties: e. g. the curves of Examples 0.1 and 0.3, and the cubic surface of Example 0.9.

Remark 1.5 (Affine varieties are zero loci of ideals)**.** Let f and g be polynomials that vanish on a certain subset $X \subset \mathbb{A}^n$. Then $f+g$ and hf for any polynomial h clearly vanish on X as well. This means that the set $S \subset K[x_1,\dots,x_n]$ defining an affine variety $X = V(S)$ is certainly not unique: for any $f,g \in S$ and any polynomial h we can add $f+g$ and hf to S without changing its zero locus. In other words, if

$$I = (S) = \{h_1 f_1 + \cdots + h_m f_m : m \in \mathbb{N}, f_1,\dots,f_m \in S, h_1,\dots,h_m \in K[x_1,\dots,x_n]\}$$

is the ideal generated by S, then $V(I) = V(S)$. Hence any affine variety in $\mathbb{A}^n$ can be written as the zero locus of an ideal in $K[x_1,\dots,x_n]$.

Example 1.6 (Affine varieties in $\mathbb{A}^1$)**.** Let X be an affine variety in $\mathbb{A}^1$. By Remark 1.5 we can then write $X = V(I)$ for an ideal $I \trianglelefteq K[x]$. But $K[x]$ is a principal ideal domain [G1, Example 10.33 (a)]. Hence we have $I = (f)$ for some polynomial $f \in K[x]$, and thus $X = V(f)$.

As zero loci of non-zero polynomials in one variable are always finite, this means that any affine variety in $\mathbb{A}^1$ not equal to $\mathbb{A}^1$ itself must be a finite set. Conversely, any finite subset $\{a_1,\dots,a_n\} = V((x-a_1)\,\cdots\,(x-a_n))$ of $\mathbb{A}^1$ is an affine variety, and thus we conclude that the affine varieties in $\mathbb{A}^1$ are exactly the finite sets and $\mathbb{A}^1$ itself.

Unfortunately, for more than one variable we cannot use a similar argument to classify the affine varieties in $\mathbb{A}^n$ as the multivariate polynomial rings $K[x_1,\dots,x_n]$ are not principal ideal domains. However, we still have the following result that we will borrow from commutative algebra.

Proposition 1.7 (**Hilbert's Basis Theorem** [G5, Proposition 7.13 and Remark 7.15])**.** *Every ideal in the polynomial ring $K[x_1,\dots,x_n]$ can be generated by finitely many elements.*

Remark 1.8 (Affine varieties are zero loci of finitely many polynomials)**.** Let $X = V(S)$ be an affine variety. Then the ideal generated by S can be written as $(S) = (f_1,\dots,f_m)$ for some $f_1,\dots,f_m \in S$ by Proposition 1.7, and hence $X = V(S) = V(f_1,\dots,f_m)$ by Remark 1.5. So every affine variety is the zero locus of finitely many polynomials.

Exercise 1.9. Prove that every affine variety $X \subset \mathbb{A}^n$ consisting of only finitely many points can be written as the zero locus of n polynomials.

(Hint: interpolation.)

There is another reason why Remark 1.5 is important: it is in some sense the basis of algebraic geometry since it relates *geometric objects* (affine varieties) to *algebraic objects* (ideals). In fact, it will be the main goal of this first chapter to make this correspondence precise. We have already assigned affine varieties to ideals in Definition 1.2 (c) and Remark 1.5, so let us now introduce an operation that does the opposite job.

Definition 1.10 (Ideal of a subset of $\mathbb{A}^n$)**.** Let $X \subset \mathbb{A}^n$ be any subset. Then

$$I(X) := \{f \in K[x_1,\dots,x_n] : f(x) = 0 \text{ for all } x \in X\}$$

is called the **ideal** of X (note that this is indeed an ideal by Remark 1.5).

Example 1.11 (Ideal of a point)**.** Let $a = (a_1,\dots,a_n) \in \mathbb{A}^n_K$ be a point. Then the ideal of the one-point set $\{a\}$ is $I(a) := I(\{a\}) = (x_1 - a_1,\dots,x_n - a_n)$:

"$\subset$" If $f \in I(a)$ then $f(a) = 0$. This means that replacing each x_i by a_i in f gives zero, i. e. that f is zero modulo $(x_1 - a_1,\dots,x_n - a_n)$. Hence $f \in (x_1 - a_1,\dots,x_n - a_n)$.

"$\supset$" If $f \in (x_1 - a_1,\dots,x_n - a_n)$ then $f = \sum_{i=1}^n (x_i - a_i)\, f_i$ for some $f_1,\dots,f_n \in K[x_1,\dots,x_n]$, and so certainly $f(a) = 0$, i. e. $f \in I(a)$.

We have now constructed operations

$$\begin{array}{ccc} \{\text{affine varieties in } \mathbb{A}^n\} & \longleftrightarrow & \{\text{ideals in } K[x_1,\dots,x_n]\} \\ X & \longmapsto & I(X) \\ V(J) & \longleftarrow & J \end{array}$$

and should check whether they actually give a *bijective* correspondence between ideals and affine varieties. The following lemma tells us the positive results in this direction. 01

Lemma 1.12. *Let S and S' be subsets of $K[x_1,\dots,x_n]$, and let X and X' be subsets of $\mathbb{A}^n$.*

(a) *If $X \subset X'$ then $I(X') \subset I(X)$.*
If $S \subset S'$ then $V(S') \subset V(S)$.
We say that the operations $V(\,\cdot\,)$ and $I(\,\cdot\,)$ reverse inclusions.

(b) *$X \subset V(I(X))$ and $S \subset I(V(S))$.*

(c) *If X is an affine variety then $V(I(X)) = X$.*

Proof.

(a) Let $X \subset X'$. If $f \in I(X')$, i. e. $f(x) = 0$ for all $x \in X'$, then certainly also $f(x) = 0$ for all $x \in X$, and hence $f \in I(X)$. The second statement follows analogously.

(b) Let $x \in X$. Then $f(x) = 0$ for every $f \in I(X)$, and thus by definition we have $x \in V(I(X))$. Again, the second inclusion follows in the same way.

(c) By (b) it suffices to prove "$\subset$". As X is an affine variety we can write $X = V(S)$ for some $S \subset K[x_1,\dots,x_n]$. Then $S \subset I(V(S))$ by (b), and thus $V(S) \supset V(I(V(S)))$ by (a). Replacing $V(S)$ by X again now gives the required inclusion. □

By this lemma, the only thing left that would be needed for a bijective correspondence between ideals and affine varieties would be $I(V(J)) \subset J$ for any ideal J (so that then $I(V(J)) = J$ by part (b)). Unfortunately, the following example shows that there are two reasons why this is not true in general.

Example 1.13 (The inclusion $J \subset I(V(J))$ is strict in general)**.**

(a) Let $J \trianglelefteq \mathbb{C}[x]$ be a non-zero ideal. As $\mathbb{C}[x]$ is a principal ideal domain [G1, Example 10.33 (a)] and $\mathbb{C}$ is algebraically closed, we must have

$$J = \big((x-a_1)^{k_1} \cdot \cdots \cdot (x-a_n)^{k_n} \big)$$

for some $n \in \mathbb{N}$, distinct $a_1,\dots,a_n \in \mathbb{C}$, and $k_1,\dots,k_n \in \mathbb{N}_{>0}$. Obviously, the zero locus of this ideal in $\mathbb{A}^1$ is $V(J) = \{a_1,\dots,a_n\}$. The polynomials vanishing on this set are precisely those that contain each factor $x - a_i$ for $i = 1,\dots,n$ at least once, i. e. we have

$$I(V(J)) = \big((x-a_1) \cdot \cdots \cdot (x-a_n) \big).$$

If at least one of the numbers $k_1,\dots,k_n$ is greater than 1, this is a bigger ideal than J. In other words, *the zero locus of an ideal does not see powers of polynomials*: as a power f^k of a polynomial f has the same zero locus as f itself, the information about this power is lost when applying the operation $I(V(\,\cdot\,))$.

(b) The situation is even worse for ground fields that are not algebraically closed: the ideal $J = (x^2+1) \trianglelefteq \mathbb{R}[x]$ has an empty zero locus in $\mathbb{A}^1$, and so we get $I(V(J)) = I(\emptyset) = \mathbb{R}[x]$. So in this case the complete information on the ideal J is lost when applying the operation $I(V(\cdot))$.

To overcome the first of these problems, we just have to restrict our attention to ideals with the property that they contain a polynomial f whenever they contain a power f^k of it. The following definition accomplishes this.

Definition 1.14 (Radicals and radical ideals)**.** Let I be an ideal in a ring R.

(a) We call

$$\sqrt{I} := \{f \in R : f^n \in I \text{ for some } n \in \mathbb{N}\}$$

the **radical** of I.

(b) The ideal I is said to be a **radical ideal** if $\sqrt{I} = I$.

Remark 1.15. Again let I be an ideal in a ring R.

(a) The radical $\sqrt{I}$ of I is always an ideal:

- We have $0 \in \sqrt{I}$, since $0 \in I$.
- If $f, g \in \sqrt{I}$, i. e. $f^n \in I$ and $g^m \in I$ for some $n, m \in \mathbb{N}$, then

$$(f+g)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} f^k g^{n+m-k}$$

is again an element of I, since in each summand we must have that the power of f is at least n (in which case $f^k \in I$) or the power of g is at least m (in which case $g^{n+m-k} \in I$). Hence $f + g \in \sqrt{I}$.

- If $h \in R$ and $f \in \sqrt{I}$, i. e. $f^n \in I$ for some $n \in \mathbb{N}$, then $(hf)^n = h^n f^n \in I$, and hence $hf \in \sqrt{I}$.

Moreover, it is obvious that $I \subset \sqrt{I}$ (we can always take $n = 1$ in Definition 1.14 (a)). Hence I is radical if and only if $\sqrt{I} \subset I$, i. e. if $f^n \in I$ for some $n \in \mathbb{N}$ implies $f \in I$.

(b) As expected from the terminology, the radical of I is a radical ideal: if $f^n \in \sqrt{I}$ for some $f \in R$ and $n \in \mathbb{N}$ then $(f^n)^m = f^{nm} \in I$ for some $m \in \mathbb{N}$, and hence $f \in \sqrt{I}$.

(c) If I is the ideal of an affine variety X then I is radical: if $f \in \sqrt{I}$ then f^k vanishes on X, hence f vanishes on X and we also have $f \in I$.

Example 1.16. Continuing Example 1.13 (a), the radical of the ideal

$$J = \big((x-a_1)^{k_1} \cdot \cdots \cdot (x-a_n)^{k_n} \big) \quad \trianglelefteq \mathbb{C}[x]$$

consists of all polynomials $f \in \mathbb{C}[x]$ such that $(x-a_1)^{k_1} \cdot \cdots \cdot (x-a_n)^{k_n}$ divides f^k for large enough k. This is obviously the set of all polynomials containing each factor $x - a_i$ for $i = 1, \dots, n$ at least once, i. e. we have

$$\sqrt{J} = \big((x-a_1) \cdot \cdots \cdot (x-a_n) \big).$$

One should note however that the explicit computation of radicals is in general hard and requires algorithms of computer algebra.

In our example at hand we therefore see that $I(V(J)) = \sqrt{J}$, resp. that $I(V(J)) = J$ if J is radical. In fact, this holds in general for ideals in polynomial rings over algebraically closed fields. This statement is usually referred to as *Hilbert's Nullstellensatz* ("theorem of the zeroes"); it is another fact that we will quote here from commutative algebra.

Proposition 1.17 (Hilbert's Nullstellensatz [G5, Corollary 10.14]). *Let K be an algebraically closed field. Then for every ideal $J \trianglelefteq K[x_1, \dots, x_n]$ we have $I(V(J)) = \sqrt{J}$. In particular, there is an inclusion-reversing one-to-one correspondence*

$$\begin{array}{rcl} \{\text{affine varieties in } \mathbb{A}^n\} & \longleftrightarrow & \{\text{radical ideals in } K[x_1,\dots,x_n]\} \\ X & \longmapsto & I(X) \\ V(J) & \longleftarrow & J. \end{array}$$

Proof. The main statement $I(V(J)) = \sqrt{J}$ is proven in [G5, Corollary 10.14]. The correspondence then follows from what we have already seen:

- $I(\,\cdot\,)$ maps affine varieties to radical ideals by Remark 1.15 (c);
- we have $V(I(X)) = X$ for any affine variety X by Lemma 1.12 (c) and $I(V(J)) = J$ for any radical ideal J by our main statement;

- the correspondence reverses inclusions by Lemma 1.12 (a). □

As we have already mentioned, this result is absolutely central for algebraic geometry since it allows us to translate geometric objects into algebraic ones. Note however that the introduction of radical ideals allowed us to solve Problem (a) in Example 1.13, but not Problem (b): for ground fields that are not algebraically closed the statement of Proposition 1.17 is clearly false since e. g. the ideal $J = (x^2+1) \trianglelefteq \mathbb{R}[x]$ is radical but has an empty zero locus, so that $I(V(J)) = \mathbb{R}[x] \neq (x^2+1) = \sqrt{J}$. Let us therefore agree:

$\boxed{\text{From now on, our ground field } K \text{ will always be assumed to be algebraically closed.}}$

Remark 1.18.

(a) Let $J \trianglelefteq K[x_1,\dots,x_n]$ be an ideal in the polynomial ring (over an algebraically closed field). If $J \neq K[x_1,\dots,x_n]$ then J has a zero, i. e. $V(J)$ is non-empty: otherwise we would have $\sqrt{J} = I(V(J)) = I(\emptyset) = K[x_1,\dots,x_n]$ by Proposition 1.17, which means $1 \in \sqrt{J}$ and gives us the contradiction $1 \in J$. This statement can be thought of as a generalization of the algebraic closure property that a non-constant univariate polynomial has a zero. It is the origin of the name "Nullstellensatz" for Proposition 1.17.

(b) Another easy consequence of Proposition 1.17 is that polynomials and polynomial functions on $\mathbb{A}^n$ agree: if $f,g \in K[x_1,\dots,x_n]$ are two polynomials defining the same function on $\mathbb{A}^n$, i. e. such that $f(x) = g(x)$ for all $x \in \mathbb{A}^n$, then

$$f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{(0)} = (0)$$

and hence $f = g$ in $K[x_1,\dots,x_n]$. So $K[x_1,\dots,x_n]$ can be thought of as the ring of polynomial functions on $\mathbb{A}^n$. Note that this is false for general fields, since e. g. the polynomial $x^2+x \in \mathbb{Z}_2[x]$ defines the zero function on $\mathbb{A}^1_{\mathbb{Z}_2}$, although it is not the zero polynomial.

More generally, if X is an affine variety then two polynomials $f,g \in K[x_1,\dots,x_n]$ define the same polynomial function on X, i. e. $f(x) = g(x)$ for all $x \in X$, if and only if $f - g \in I(X)$. So the quotient ring $K[x_1,\dots,x_n]/I(X)$ can be thought of as the ring of polynomial functions on X. Let us make this into a precise definition.

Definition 1.19 (Polynomial functions and coordinate rings)**.** Let $X \subset \mathbb{A}^n$ be an affine variety. A **polynomial function** on X is a map $X \to K$ that is of the form $x \mapsto f(x)$ for some $f \in K[x_1,\dots,x_n]$. By Remark 1.18 (b) the ring of all polynomial functions on X is just the quotient ring

$$A(X) := K[x_1,\dots,x_n]/I(X).$$

It is usually called the **coordinate ring** of the affine variety X.

According to this definition, we can think of the elements of $A(X)$ in the following both as functions on X and as elements of the quotient ring $K[x_1,\dots,x_n]/I(X)$. We can use this ring to define the concepts introduced so far in a relative setting, i. e. consider zero loci of ideals in $A(Y)$ and varieties contained in Y for a fixed ambient affine variety Y that is not necessarily $\mathbb{A}^n$:

Construction 1.20 (Relative version of the correspondence between varieties and radical ideals)**.** Let $Y \subset \mathbb{A}^n$ be an affine variety. The following two constructions are then completely analogous to those in Definitions 1.2 (c) and 1.10:

(a) For a subset $S \subset A(Y)$ of polynomial functions on Y we define its **zero locus** as

$$V(S) := V_Y(S) := \{x \in Y : f(x) = 0 \text{ for all } f \in S\} \quad \subset Y.$$

The subsets that are of this form are obviously precisely the affine varieties contained in X. They are called **affine subvarieties** of Y.

(b) For a subset $X \subset Y$ the **ideal** of X in Y is defined to be

$$I(X) := I_Y(X) := \{f \in A(Y) : f(x) = 0 \text{ for all } x \in X\} \quad \trianglelefteq A(Y).$$

With the same arguments as above, all results considered so far then hold in this relative setting as well. Let us summarize them here again:

Proposition 1.21. *Let Y be an affine variety in $\mathbb{A}^n$.*

(a) *(**Hilbert's Basis Theorem**) Every ideal in $A(Y)$ can be generated by finitely many elements.*

(b) *(**Hilbert's Nullstellensatz**) For any ideal $J \trianglelefteq A(Y)$ we have $I_Y(V_Y(J)) = \sqrt{J}$. In particular, there is an inclusion-reversing one-to-one correspondence*

$$\begin{aligned} \{\text{affine subvarieties of } Y\} &\longleftrightarrow \{\text{radical ideals in } A(Y)\} \\ X &\longmapsto I_Y(X) \\ V_Y(J) &\longleftarrow\!\shortmid J. \end{aligned}$$

(c) *For a subvariety X of Y we have $A(X) \cong A(Y)/I_Y(X)$.*

Proof. As in our earlier version, the proof of (a) is covered by [G5, Proposition 7.13 and Remark 7.15], the proof of (b) by [G5, Corollary 10.14] and Proposition 1.17. The statement (c) follows in the same way as in Remark 1.18 (b). □

Exercise 1.22. Determine the radical of the ideal $(x_1^3 - x_2^6, x_1x_2 - x_2^3) \trianglelefteq \mathbb{C}[x_1,x_2]$.

Exercise 1.23. Let X be an affine variety. Show that the coordinate ring $A(X)$ is a field if and only if X is a single point.

In the rest of this chapter we want to study the basic properties of the operations $V(\cdot)$ and $I(\cdot)$.

Lemma 1.24 (Properties of $V(\cdot)$)**.** *Let X be an affine variety.*

(a) *If J is any index set and $\{S_i : i \in J\}$ a family of subsets of $A(X)$ then $\bigcap_{i\in J} V(S_i) = V(\bigcup_{i\in J} S_i)$ in X.*

(b) *For $S_1, S_2 \subset A(X)$ we have $V(S_1) \cup V(S_2) = V(S_1S_2)$ in X, where as usual we set $S_1S_2 := \{fg : f \in S_1, g \in S_2\}$.*

In particular, arbitrary intersections and finite unions of affine subvarieties of X are again affine subvarieties of X.

Proof.

(a) We have $x \in \bigcap_{i\in J} V(S_i)$ if and only if $f(x) = 0$ for all $f \in S_i$ for all $i \in J$, which is the case if and only if $x \in V(\bigcup_{i\in J} S_i)$.

(b) "$\subset$" If $x \in V(S_1) \cup V(S_2)$ then $f(x) = 0$ for all $f \in S_1$ or $g(x) = 0$ for all $g \in S_2$. In any case this means that $(fg)(x) = 0$ for all $f \in S_1$ and $g \in S_2$, i. e. that $x \in V(S_1S_2)$.

"$\supset$" If $x \notin V(S_1) \cup V(S_2)$, i. e. $x \notin V(S_1)$ and $x \notin V(S_2)$, then there are $f \in S_1$ and $g \in S_2$ with $f(x) \neq 0$ and $g(x) \neq 0$. Then $(fg)(x) \neq 0$, and hence $x \notin V(S_1S_2)$. □

Remark 1.25 (Ideal-theoretic version of the properties of $V(\cdot)$)**.** If we want to consider zero loci of ideals rather than of general subsets of $A(X)$, then the properties of Lemma 1.24 take a slightly different form. To see this, let J_1 and J_2 be any ideals in $A(X)$.

(a) The ideal generated by $J_1 \cup J_2$ is just the *sum of ideals* $J_1 + J_2 = \{f + g : f \in J_1, g \in J_2\}$. So with Remark 1.5 the result of Lemma 1.24 (a) translates into

$$V(J_1) \cap V(J_2) = V(J_1 + J_2).$$

(b) In the same way as in (a), Lemma 1.24 (b) implies that $V(J_1) \cup V(J_2)$ is equal to the zero locus of the ideal generated by J_1J_2. Unfortunately, the usual convention is that for two *ideals* J_1 and J_2 (instead of arbitrary sets) the notation J_1J_2 denotes the ideal generated by all products fg with $f \in J_1$ and $g \in J_2$, which is called the *product of the ideals* J_1 and J_2 — rather than the set of all such products fg itself. So we get

$$V(J_1) \cup V(J_2) = V(J_1J_2)$$

with this modified definition of the product $J_1 J_2$.

(c) Another common operation on ideals is the *intersection* $J_1 \cap J_2$. In general, this ideal is different from the ones considered above, but we can show that there is always the relation
$$\sqrt{J_1 \cap J_2} = \sqrt{J_1 J_2}:$$
"$\subset$" If $f \in \sqrt{J_1 \cap J_2}$ then $f^n \in J_1 \cap J_2$ for some n. This means that $f^{2n} = f^n \cdot f^n \in J_1 J_2$, and hence that $f \in \sqrt{J_1 J_2}$.

"$\supset$" For $f \in \sqrt{J_1 J_2}$ we have $f^n \in J_1 J_2$ for some n. Then $f^n \in J_1 \cap J_2$, and thus $f \in \sqrt{J_1 \cap J_2}$.

By Proposition 1.21 (b) this means that $I(V(J_1 \cap J_2)) = I(V(J_1 J_2))$, and hence by applying $V(\cdot)$ that
$$V(J_1 \cap J_2) = V(J_1 J_2) = V(J_1) \cup V(J_2)$$
by (b).

Finally, for completeness let us also formulate the properties of Lemma 1.24 and Remark 1.25 in terms of the operation $I(\cdot)$ rather than $V(\cdot)$.

Lemma 1.26 (Properties of $I(\cdot)$)**.** *Let X be an affine variety, and let Y_1 and Y_2 be affine subvarieties of X. Then:*

(a) $I(Y_1 \cap Y_2) = \sqrt{I(Y_1) + I(Y_2)}$;

(b) $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

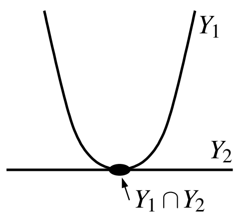
Proof.

(a) We have
$$\begin{aligned} I(Y_1 \cap Y_2) &= I(V(I(Y_1)) \cap V(I(Y_2))) && \text{(Proposition 1.21 (b))} \\ &= I(V(I(Y_1) + I(Y_2))) && \text{(Remark 1.25 (a))} \\ &= \sqrt{I(Y_1) + I(Y_2)}. && \text{(Proposition 1.21 (b))} \end{aligned}$$

(b) A polynomial function $f \in A(X)$ is contained in $I(Y_1 \cup Y_2)$ if and only if $f(x) = 0$ for all $x \in Y_1$ and all $x \in Y_2$, which is the case if and only if $f \in I(Y_1) \cap I(Y_2)$. □ 02

Remark 1.27. Recall from Remark 1.15 (c) that ideals of affine varieties are always radical. So in particular, Lemma 1.26 (b) shows that intersections of radical ideals in $A(X)$ are again radical — which could of course also be checked directly. In contrast, sums of radical ideals are in general not radical, and hence taking the radical in Lemma 1.26 (a) is really necessary.

In fact, there is also a geometric interpretation behind this fact. Consider for example the affine varieties $Y_1, Y_2 \subset \mathbb{A}^1_{\mathbb{C}}$ with ideals $I(Y_1) = (x_2 - x_1^2)$ and $I(Y_2) = (x_2)$ whose real points are shown in the picture on the right. Their intersection $Y_1 \cap Y_2$ is obviously the origin with ideal $I(Y_1 \cap Y_2) = (x_1, x_2)$. But
$$I(Y_1) + I(Y_2) = (x_2 - x_1^2, x_2) = (x_1^2, x_2)$$
is not a radical ideal; only its radical is equal to $I(Y_1 \cap Y_2) = (x_1, x_2)$.



The geometric meaning of the non-radical ideal $I(Y_1) + I(Y_2) = (x_1^2, x_2)$ is that Y_1 and Y_2 are tangent at the intersection point: if we consider the function $x_2 - x_1^2$ defining Y_1 on the x_1-axis Y_2 (where it is equal to $-x_1^2$) we see that it vanishes to order 2 at the origin. This means that Y_1 and Y_2 share the x_1-axis as common tangent direction, so that the intersection $Y_1 \cap Y_2$ can be thought of as "extending to an infinitesimally small amount in the x_1-direction", and we can consider Y_1 and Y_2 as "intersecting with multiplicity 2" at the origin. We will see later in Definition 12.23 (b) how such intersection multiplicities can be defined rigorously.