9. Intersection theory

A k-cycle on a scheme X (that is always assumed to be separated and of finite type over an algebraically closed field in this section) is a finite formal linear combination $\sum_i n_i[V_i]$ with $n_i \in \mathbb{Z}$, where the V_i are k-dimensional subvarieties of X. The group of k-cycles is denoted $Z_k(X)$. A rational function φ on any subvariety $Y \subset X$ of dimension $k+1$ determines a cycle $\operatorname{div}(\varphi) \in Z_k(X)$, which is just the zeroes of φ minus the poles of φ, counted with appropriate multiplicities. The subgroup $B_k(X) \subset Z_k(X)$ generated by cycles of this form is called the group of k-cycles that are rationally equivalent to zero. The quotient groups $A_k(X) = Z_k(X)/B_k(X)$ are the groups of cycle classes or Chow groups. They are the main objects of study in intersection theory. The Chow groups of a scheme should be thought of as being analogous to the homology groups of a topological space.

A morphism $f : X \to Y$ is called proper if inverse images of compact sets (in the classical topology) are compact. Any proper morphism f gives rise to push-forward homomorphisms $f_* : A_*(X) \to A_*(Y)$ between the Chow groups. On the other hand, some other morphisms $f : X \to Y$ (e. g. inclusions of open subsets or projections from vector bundles) admit pull-back maps $f^* : A_*(Y) \to A_*(X)$.

If X is a purely n-dimensional scheme, a Weil divisor is an element of $Z_{n-1}(X)$. In contrast, a Cartier divisor is a global section of the sheaf $\mathcal{K}_X^* / O_X^*$. Every Cartier divisor determines a Weil divisor. On smooth schemes, Cartier and Weil divisors agree. On almost any scheme, Cartier divisors modulo linear equivalence correspond exactly to line bundles.

We construct bilinear maps $\operatorname{Pic} X \times A_k(X) \to A_{k-1}(X)$ that correspond geometrically to taking intersections of the divisor (a codimension-1 subset of X) with the k-dimensional subvariety. If one knows the Chow groups of a space and the above intersection products, one arrives at Bézout style theorems that allow to compute the number of intersection points of k divisors on X with a k-dimensional subspace.

9.1. **Chow groups.** Having discussed the basics of scheme theory, we will now start with the foundations of intersection theory. The idea of intersection theory is the same as that of homology in algebraic topology. Roughly speaking, what one does in algebraic topology is to take e. g. a real differentiable manifold X of dimension n and an integer $k \geq 0$, and consider formal linear combinations of real k-dimensional submanifolds (with boundary) on X with integer coefficients, called cycles. If $Z_k(X)$ is the group of closed cycles (those having no boundary) and $B_k(X) \subset Z_k(X)$ is the group of those cycles that are boundaries of $(k+1)$-dimensional cycles, then the homology group $H_k(X, \mathbb{Z})$ is the quotient $Z_k(X)/B_k(X)$.

There are (at least) two main applications of this. First of all, the groups $H_k(X, \mathbb{Z})$ are (in contrast to the $Z_k(X)$ and $B_k(X)$) often finitely generated groups and provide invariants of the manifold X that can be used for classification purposes. Secondly, there are intersection products: homology classes in $H_{n-k}(X, \mathbb{Z})$ and $H_{n-l}(X, \mathbb{Z})$ can be "multiplied" to give a class in $H_{n-k-l}(X, \mathbb{Z})$ that geometrically corresponds to taking intersections of submanifolds. Hence if we are for example given submanifolds V_i of X whose codimensions sum up to n (so that we expect a finite number of points in the intersection $\bigcap_i V_i$), then this number can often be computed easily by taking the corresponding products in homology.

Our goal is to establish a similar theory for schemes. For any scheme of finite type over a ground field and any integer $k \geq 0$ we will define the so-called Chow groups $A_k(X)$ whose elements are formal linear combinations of k-dimensional closed subvarieties of X, modulo "boundaries" in a suitable sense. The formal properties of these groups $A_k(X)$ will be similar to those of homology groups; if the ground field is $\mathbb{C}$ you might even want to think of the $A_k(X)$ as being "something like" $H_{2k}(X, \mathbb{Z})$, although these groups are usually different. But there is always a map $A_k(X) \to H_{2k}(X, \mathbb{Z})$ (at least if one uses the "right" homology theory, see [F] chapter 19 for details), so you can think of elements in the Chow

groups as something that determines a homology class, but this map is in general neither injective nor surjective.

Another motivation for the Chow groups $A_k(X)$ is that they generalize our notions of divisors and divisor classes. In fact, if X is a smooth projective curve then $A_0(X)$ will be by definition the same as $\text{Pic}\,X$. In general, the definition of the groups $A_k(X)$ is very similar to our definition of divisors: we consider the free Abelian groups $Z_k(X)$ generated by the k-dimensional subvarieties of X. There is then a subgroup $B_k(X) \subset Z_k(X)$ that corresponds to those linear combinations of subvarieties that are zeros minus poles of rational functions. The Chow groups are then the quotients $A_k(X) = Z_k(X)/B_k(X)$.

To make sense of this definition, the first thing we have to do is to define the divisor of a rational function (see definition 6.3.4) in the higher-dimensional case. This is essentially a problem of commutative algebra, so we will only sketch it here. The important ingredient is the notion of the length of a module.

Remark 9.1.1. (For the following facts we refer to [AM] chapter 6 and [F] section A.1.) Let M be a finitely generated module over a Noetherian ring R. Then there is a so-called **composition series**, i. e. a finite chain of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M \qquad (*)$$

such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some prime ideals $\mathfrak{p}_i \in R$. The series is not unique, but for any prime ideal $\mathfrak{p} \subset R$ the number of times $\mathfrak{p}$ occurs among the $\mathfrak{p}_i$ does not depend on the series.

The geometric meaning of this composition series is easiest explained in the case where R is an integral domain and $M = R/I$ for some ideal $I \subset R$. In this case $\text{Spec}\,M$ is a closed subscheme of the irreducible scheme $\text{Spec}\,R$ (see examples 5.2.3 and 7.2.10). The prime ideals $\mathfrak{p}_i$ are then precisely the ideals of the irreducible (and maybe embedded) components of $\text{Spec}\,M$, or in other words the prime ideals associated to all primary ideals in the primary decomposition of I. The number of times $\mathfrak{p}$ occurs among the $\mathfrak{p}_i$ can be thought of as the "multiplicity" of the corresponding component in the scheme. For example, if I is a radical ideal (so $\text{Spec}\,M$ is reduced) then the $\mathfrak{p}_i$ are precisely the ideals of the irreducible components of $\text{Spec}\,M$, all occurring once.

We will need this construction mainly in the case where $I = (f) \subset R$ is the ideal generated by a single (non-zero) function. In this case all irreducible components of $\text{Spec}\,M$ have codimension 1. If $\mathfrak{p} \subset R$ is a prime ideal corresponding to any codimension-1 subvariety of $\text{Spec}\,R$ we can consider a composition series as above for the localized module $M_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$. As the only prime ideals in $R_{\mathfrak{p}}$ are (0) and $\mathfrak{p}R_{\mathfrak{p}}$ (corresponding geometrically to $\text{Spec}\,R$ and $\text{Spec}\,M$, respectively) and f does not vanish identically on $\text{Spec}\,M$, the only prime ideal that can occur in the composition series of $M_{\mathfrak{p}}$ is $\mathfrak{p}R_{\mathfrak{p}}$. The number of times it occurs, i. e. the length r of the composition series, is then called the **length** of the module $M_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$, denoted $l_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. It is equal to the number of times $\mathfrak{p}$ occurs in the composition series of M over R. By what we have said above, we can interpret this number geometrically as the multiplicity of the subvariety corresponding to $\mathfrak{p}$ in the scheme $\text{Spec}\,R/(f)$, or in other words as the order of vanishing of f at this codimension-1 subvariety.

We should rephrase these ideas in terms of general (not necessarily affine) schemes. So let X be a scheme, and let $V \subset X$ be a subvariety of codimension 1. Note that V can be considered as a *point* in the scheme X, so it makes sense to talk about the stalk $O_{X,V}$ of the structure sheaf O_X at V. If $U = \text{Spec}\,R \subset X$ is any affine open subset with non-empty intersection with V then $O_{X,V}$ is just the localized ring $R_{\mathfrak{p}}$ where $\mathfrak{p}$ is the prime ideal corresponding to the subvariety $V \cap U$ of U (see proposition 5.1.12 (i)). So if $f \in O_{X,V}$ is a local function around V then its order of vanishing at the codimension-1 subvariety V is simply the length $l_{O_{X,V}}(O_{X,V}/(f))$. To define the order of a possibly rational function φ on X we just have to observe that the field of fractions of the ring $O_{X,V}$ is equal to the field of

rational functions on X. So we can write φ as $\frac{f}{g}$ for some $f, g \in O_{X,V}$ and simply define the order of φ at V to be the difference of the orders of f and g at V.

With these prerequisites we can now define the Chow groups in complete analogy to the Picard group of divisor classes in section 6.3. For the rest of this section by a scheme we will always mean a scheme of finite type over some algebraically closed field (that is not necessarily smooth, irreducible, or reduced). A variety is a reduced and irreducible (but not necessarily smooth) scheme.

Definition 9.1.2. Let X be a variety, and let $V \subset X$ be a subvariety of codimension 1, and set $R = O_{X,V}$. For every non-zero $f \in R \subset K(X)$ we define the **order** of f at V to be the integer $\operatorname{ord}_V(f) := l_R(R/(f))$. If $\varphi \in K(X)$ is a non-zero rational function we write $\varphi = \frac{f}{g}$ with $f, g \in R$ and define the order of φ at V to be

$$\operatorname{ord}_V(\varphi) := \operatorname{ord}_V(f) - \operatorname{ord}_V(g).$$

To show that this is well-defined, i.e. that $\operatorname{ord}_V \frac{f}{g} = \operatorname{ord}_V \frac{f'}{g'}$ whenever $fg' = gf'$, one uses the exact sequence

$$0 \to R/(a) \xrightarrow{\cdot b} R/(ab) \to R/(b) \to 0$$

and the fact that the length of modules is additive on exact sequences. From this it also follows that the order function is a homomorphism of groups $\operatorname{ord}_V : K(X)^* := K(X) \setminus \{0\} \to \mathbb{Z}$.

Example 9.1.3. Let $X = \mathbb{A}^1 = \operatorname{Spec} k[x]$ and let $V = \{0\} \subset X$ be the origin. Consider the function $\varphi = x^r$ for $r \geq 0$. Then $R = O_{X,V} = k[x]_{(x)}$, and $R/(x) \cong k$. So as $R/(x^r) = \{a_0 + a_1 x + \cdots + a_{r-1} x^{r-1}\}$ has vector space dimension r over k we conclude that $\operatorname{ord}_0(x^r) = r$, as expected. By definition, we then have the equality $\operatorname{ord}_0(x^r) = r$ for all $r \in \mathbb{Z}$.

Definition 9.1.4. Let X be a scheme. For $k \geq 0$ denote by $Z_k(X)$ the free Abelian group generated by the k-dimensional subvarieties of X. In other words, the elements of $Z_k(X)$ are finite formal sums $\sum_i n_i [V_i]$, where $n_i \in \mathbb{Z}$ and the V_i are k-dimensional (closed) subvarieties of X. The elements of $Z_k(X)$ are called **cycles** of dimension k.

For any $(k+1)$-dimensional subvariety W of X and any non-zero rational function φ on W we define a cycle of dimension k on X by

$$\operatorname{div}(\varphi) = \sum_V \operatorname{ord}_V(\varphi)[V] \in Z_k(X),$$

called the **divisor** of φ, where the sum is taken over all codimension-1 subvarieties V of W. Note that this sum is always finite: it suffices to check this on a finite affine open cover $\{U_i\}$ of W and for $\varphi \in O_{U_i}(U_i)$, where it is obvious as $Z(\varphi)$ is closed and U_i is Noetherian.

Let $B_k(X) \subset Z_k(X)$ be the subgroup generated by all cycles of the form $\operatorname{div}(\varphi)$ for all $W \subset X$ and $\varphi \in K(W)^*$ as above. We define the group of k-dimensional **cycle classes** to be the quotient $A_k(X) = Z_k(X)/B_k(X)$. These groups are usually called the **Chow groups** of X. Two cycles in $Z_k(X)$ that determine the same element in $A_k(X)$ are said to be **rationally equivalent**.

We set $Z_*(X) = \bigoplus_{k \geq 0} Z_k(X)$ and $A_*(X) = \bigoplus_{k \geq 0} A_k(X)$.

Example 9.1.5. Let X be a scheme of pure dimension n. Then $B_n(X)$ is trivially zero, and thus $A_n(X) = Z_n(X)$ is the free Abelian group generated by the irreducible components of X. In particular, if X is an n-dimensional variety then $A_n(X) \cong \mathbb{Z}$ with $[X]$ as a generator. In the same way, $Z_k(X)$ and $A_k(X)$ are trivially zero if $k > n$.

Example 9.1.6. Let X be a smooth projective curve. Then $Z_0(X) = \operatorname{Div} X$ and $A_0(X) = \operatorname{Pic} X$ by definition. In fact, the 1-dimensional subvariety W of X in definition 9.1.4 can only be X itself, so we arrive at precisely the same definition as in section 6.3.

Example 9.1.7. Let $X = \{x_1 x_2 = 0\} \subset \mathbb{P}^2$ be the union of two projective lines $X = X_1 \cup X_2$ that meet in a point. Then $A_1(X) = \mathbb{Z}[X_1] \oplus \mathbb{Z}[X_2]$ by example 9.1.5. Moreover, $A_0(X) \cong \mathbb{Z}$ is generated by the class of any point in X. In fact, any two points on X_1 are rationally equivalent by example 9.1.6, and the same is true for X_2. As both X_1 and X_2 contain the intersection point $X_1 \cap X_2$ we conclude that all points in X are rationally equivalent. So $A_0(X) \cong \mathbb{Z}$.

Now let $P_1 \in X_1 \setminus X_2$ and $P_2 \in X_2 \setminus X_1$ be two points. Note that the line bundles $O_X(P_1)$ and $O_X(P_2)$ (defined in the obvious way: $O_X(P_i)$ is the sheaf of rational functions that are regular away from P_i and have at most a simple pole at P_i) are *not* isomorphic: if $i : X_1 \to X$ is the inclusion map of the first component, then $i^* O_X(P_1) \cong O_{\mathbb{P}^1}(1)$, whereas $i^* O_X(P_2) \cong O_{\mathbb{P}^1}$. So we see that for singular curves the one-to-one correspondence between $A_0(X)$ and line bundles no longer holds.

Example 9.1.8. Let $X = \mathbb{A}^n$. We claim that $A_0(X) = 0$. In fact, if $P \in X$ is any point, pick a line $W \cong \mathbb{A}^1 \subset \mathbb{A}^n$ through P and a linear function φ on W that vanishes precisely at P. Then $\mathrm{div}(\varphi) = [P]$. It follows that the class of any point is zero in $A_0(X)$. Therefore $A_0(X) = 0$.

Example 9.1.9. Now let $X = \mathbb{P}^n$; we claim that $A_0(X) \cong \mathbb{Z}$. In fact, if P and Q are any two distinct points in X let $W \cong \mathbb{P}^1 \subset \mathbb{P}^n$ be the line through P and Q, and let φ be a rational function on W that has a simple zero at P and a simple pole at Q. Then $\mathrm{div}(\varphi) = [P] - [Q]$, i.e. the classes in $A_0(X)$ of any two points in X are the same. It follows that $A_0(X)$ is generated by the class $[P]$ of any point in X.

On the other hand, if $W \subset X = \mathbb{P}^n$ is any curve and φ a rational function on W then we have seen in remark 6.3.5 that the degree of the divisor of φ is always zero. It follows that the class $n \cdot [P] \in A_0(X)$ for $n \in \mathbb{Z}$ can only be zero if $n = 0$. We conclude that $A_0(X) \cong \mathbb{Z}$ with the class of any point as a generator.

Example 9.1.10. Let X be a scheme, and let $Y \subset X$ be a closed subscheme with inclusion morphism $i : Y \to X$. Then there are canonical **push-forward** maps $i_* : A_k(Y) \to A_k(X)$ for any k, given by $[Z] \mapsto [Z]$ for any k-dimensional subvariety $Z \subset Y$. It is obvious from the definitions that this respects rational equivalence.

Example 9.1.11. Let X be a scheme, and let $U \subset X$ be an open subset with inclusion morphism $i : U \to X$. Then there are canonical **pull-back** maps $i^* : A_k(X) \to A_k(U)$ for any k, given by $[Z] \mapsto [Z \cap U]$ for any k-dimensional subvariety $Z \subset X$. This respects rational equivalence as $i^* \mathrm{div}(\varphi) = \mathrm{div}(\varphi|_U)$ for any rational function φ on a subvariety of X.

Remark 9.1.12. If $f : X \to Y$ is any morphism of schemes it is an important part of intersection theory to study whether there are push-forward maps $f_* : A_*(X) \to A_*(Y)$ or pull-back maps $f^* : A_*(Y) \to A_*(X)$ and which properties they have. We have just seen two easy examples of this. Note that neither example can be reversed (at least not in an obvious way):

  (i) if $Y \subset X$ is a closed subset, then a subvariety of X is in general not a subvariety of Y, so there is no pull-back morphism $A_*(X) \to A_*(Y)$ sending $[V]$ to $[V]$ for any subvariety $V \subset X$.
  (ii) if $U \subset X$ is an open subset, there are no push-forward maps $A_*(U) \to A_*(X)$: if $U = \mathbb{A}^1$ and $X = \mathbb{P}^1$ then the class of a point is zero in $A_*(U)$ but non-zero in $A_*(\mathbb{P}^1)$ by examples 9.1.8 and 9.1.9.

We will construct more general examples of push-forward maps in section 9.2, and more general examples of pull-back maps in proposition 9.1.14.

Lemma 9.1.13. *Let X be a scheme, let $Y \subset X$ be a closed subset, and let $U = X \backslash Y$. Denote the inclusion maps by $i : Y \to X$ and $j : U \to X$. Then the sequence*

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0$$

is exact for all $k \geq 0$. The homomorphism i_* is in general not injective however.

Proof. This follows more or less from the definitions. If $Z \subset U$ is any k-dimensional subvariety then the closure $\bar{Z}$ of Z in X is a k-dimensional subvariety of X with $j^*[\bar{Z}] = [Z]$. So j^* is surjective.

If $Z \subset Y$ then $Z \cap U = 0$, so $j^* \circ i_* = 0$. Conversely, assume that we have a cycle $\sum a_r[V_r] \in A_k(X)$ whose image in $A_k(U)$ is zero. This means that there are rational functions φ_s on $(k+1)$-dimensional subvarieties W_s of U such that $\sum \mathrm{div}(\varphi_s) = \sum a_r[V_r \cap U]$ on U. Now the φ_s are also rational functions on the closures of W_s in X, and as such their divisors can only differ from the old ones by subvarieties V_r' that are contained in $X \backslash U = Y$. We conclude that $\sum \mathrm{div}(\varphi_s) = \sum a_r[V_r] - \sum b_r[V_r']$ on X for some b_r. So $\sum a_r[V_r] = i_* \sum b_r[V_r']$.

As an example that i_* is in general not injective let Y be a smooth cubic curve in $X = \mathbb{P}^2$. If P and Q are two distinct points on Y then $[P] - [Q] \neq 0 \in A_0(Y) = \mathrm{Pic}\, X$ by proposition 6.3.13, but $[P] - [Q] = 0 \in A_0(X) \cong \mathbb{Z}$ by example 9.1.9. $\qquad\square$

Proposition 9.1.14. *Let X be a scheme, and let $\pi : E \to X$ be a vector bundle of rank r on X (see remark 7.3.2). Then for all $k \geq 0$ there is a well-defined surjective pull-back homomorphism $\pi^* : A_k(X) \to A_{k+r}(E)$ given on cycles by $\pi^*[V] = [\pi^{-1}(V)]$.*

Proof. It is clear that π^* is well-defined: it obviously maps k-dimensional cycles to $(k+r)$-dimensional cycles, and $\pi^* \mathrm{div}(\varphi) = \mathrm{div}(\pi^*\varphi)$ for any rational function φ on a $(k+1)$-dimensional subvariety of X.

We will prove the surjectivity by induction on $\dim X$. Let $U \subset X$ be an affine open subset over which E is of the form $U \times \mathbb{A}^r$, and let $Y = X \backslash U$. By lemma 9.1.13 there is a commutative diagram

$$
\begin{array}{ccccccc}
A_k(Y) & \longrightarrow & A_k(X) & \longrightarrow & A_k(U) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \pi^*} & & \downarrow & & \\
A_{k+r}(E|_Y) & \longrightarrow & A_{k+r}(E) & \longrightarrow & A_{k+r}(U \times \mathbb{A}^r) & \longrightarrow & 0
\end{array}
$$

with exact rows. A diagram chase (similar to that of the proof of lemma 8.2.2) shows that in order for π^* to be surjective it suffices to prove that the left and right vertical arrows are surjective. But the left vertical arrow is surjective by the induction assumption since $\dim Y < \dim X$. So we only have to show that the right vertical arrow is surjective. In other words, we have reduced to the case where $X = \mathrm{Spec}\, R$ is affine and $E = X \times \mathbb{A}^r$ is the trivial bundle. As π then factors as a sequence

$$E = X \times \mathbb{A}^r \to X \times \mathbb{A}^{r-1} \to \cdots \to X \times \mathbb{A}^1 \to X$$

we can furthermore assume that $r = 1$, so that $E = X \times \mathbb{A}^1 = \mathrm{Spec}\, R[t]$.

We have to show that $\pi^* : A_k(X) \to A_k(X \times \mathbb{A}^1)$ is surjective. So let $V \subset X \times \mathbb{A}^1$ be a $(k+1)$-dimensional subvariety, and let $W = \overline{\pi(V)}$. There are now two cases to consider:

- $\dim W = k$. Then $V = W \times \mathbb{A}^1$, so $[V] = \pi^*[W]$.
- $\dim W = k+1$. As it suffices to show that $[V]$ is in the image of the pull-back map $A_k(W) \to A_{k+r}(W \times \mathbb{A}^1)$ we can assume that $W = X$. Consider the ideal $I(V) \otimes_R K \subset K[t]$, where $K = K(W)$ denotes the quotient field of R. It is not the unit ideal as otherwise we would be in case (i). On the other hand $K[t]$ is a principal ideal domain, so $I(V) \otimes_R K$ is generated by a single polynomial $\varphi \in K[t]$.

Considering φ as a rational function on $X \times \mathbb{A}^1$ we see that the divisor of φ is precisely $[V]$ by construction, plus maybe terms of the form $\sum a_i \pi^*[W_i]$ for some $W_i \subset X$ corresponding to our tensoring with the field of rational functions $K(X)$. So $[V] = \pi^*(\sum a_i[W_i])$ (plus the divisor of a rational function), i.e. $[V]$ is in the image of π^*.

$\square$

Remark 9.1.15. Note that the surjectivity part of proposition 9.1.14 is obviously false on the cycle level, i.e. for the pull-back maps $Z_k(X) \to Z_k(E)$: not every subvariety of a vector bundle E over X is the inverse image of a subvariety in X. So this proposition is an example of the fact that working with Chow groups (instead of with the subvarieties themselves) often makes life a little easier. In fact one can show (see [F] theorem 3.3 (a)) that the pull-back maps $\pi^* : A_k(X) \to A_{k+r}(E)$ are always isomorphisms.

Corollary 9.1.16. *The Chow groups of affine spaces are given by*

$$A_k(\mathbb{A}^n) = \begin{cases} \mathbb{Z} & \text{for } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The statement for $k \geq n$ follows from example 9.1.5. For $k < n$ note that the homomorphism $A_0(\mathbb{A}^{n-k}) \to A_k(\mathbb{A}^n)$ is surjective by proposition 9.1.14, so the statement of the corollary follows from example 9.1.8. $\square$

Corollary 9.1.17. *The Chow groups of projective spaces are $A_k(\mathbb{P}^n) \cong \mathbb{Z}$ for all $0 \leq k \leq n$, with an isomorphism given by $[V] \mapsto \deg V$ for all k-dimensional subvarieties $V \subset \mathbb{P}^n$.*

Proof. The statement for $k \geq n$ follows again from example 9.1.5, so let us assume that $k < n$. We prove the statement by induction on n. By lemma 9.1.13 there is an exact sequence

$$A_k(\mathbb{P}^{n-1}) \to A_k(\mathbb{P}^n) \to A_k(\mathbb{A}^n) \to 0.$$

We have $A_k(\mathbb{A}^n) = 0$ by corollary 9.1.16, so we conclude that $A_k(\mathbb{P}^{n-1}) \to A_k(\mathbb{P}^n)$ is surjective. By the induction hypothesis this means that $A_k(\mathbb{P}^n)$ is generated by the class of a k-dimensional linear subspace. As the morphism $Z_k(\mathbb{P}^{n-1}) \to Z_k(\mathbb{P}^n)$ trivially preserves degrees it only remains to be shown that any cycle $\sum a_i[V_i]$ that is zero in $A_k(\mathbb{P}^n)$ must satisfy $\sum a_i \deg V_i = 0$. But this is clear from Bézouts theorem, as $\deg \operatorname{div}(\varphi) = 0$ for all rational functions on any subvariety of $\mathbb{P}^n$. $\square$

Remark 9.1.18. There is a generalization of corollary 9.1.17 as follows. Let X be a scheme that has a stratification by affine spaces, i.e. X has a filtration by closed subschemes $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$ such that $X_k \backslash X_{k-1}$ is a disjoint union of a_k affine spaces $\mathbb{A}^k$ for all k. For example, $X = \mathbb{P}^n$ has such a stratification with $a_k = 1$ for $0 \leq k \leq n$, namely $\emptyset \subset \mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n = X$.

We claim that then $A_k(X)$ is isomorphic to $\mathbb{Z}^{a_k}$ modulo some (possibly trivial) subgroup, where $\mathbb{Z}^{a_k}$ is generated by the classes of the closures of the a_k copies of $\mathbb{A}^k$ mentioned above. We will prove this by induction on $\dim X$, the case of dimension 0 being obvious. In fact, consider the exact sequence of lemma 9.1.13

$$A_k(X_{n-1}) \to A_k(X) \to \oplus_{i=1}^{a_n} A_k(\mathbb{A}^n) \to 0.$$

Note that X_{n-1} itself is a scheme with a filtration $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_{n-1}$ as above. So it follows that:

(i) For $k < n$ we have $A_k(\mathbb{A}^n) = 0$, so $A_k(X)$ is generated by $A_k(X_{n-1})$. Hence the claim follows from the induction hypothesis.

(ii) For $k \geq n$ we have $A_k(X_{n-1}) = 0$, so $A_n(X) \cong \oplus_{i=1}^{a_n} A_k(\mathbb{A}^n)$ is generated by the classes of the closures of the a_n copies of $\mathbb{A}^n$ in $X \backslash X_{n-1}$.

This proves the claim. In fact, one can show that $A_k(X)$ is always isomorphic to $\mathbb{Z}^{a_k}$ if X has a stratification by affine spaces as above (see [F] example 1.9.1).

In particular, this construction can be applied to compute the Chow groups of products of projective spaces and Grassmannian varieties (see exercise 3.5.4).

Remark 9.1.19. Using Chow groups, Bézout's theorem can obviously be restated as follows: we have seen in corollary 9.1.17 that $A_k(\mathbb{P}^n) \cong \mathbb{Z}$ for all $k \leq n$, with the class of a k-dimensional linear subspace as a generator. Using this isomorphism, define a product map

$$A_{n-k}(\mathbb{P}^n) \times A_{n-l}(\mathbb{P}^n) \to A_{n-k-l}(\mathbb{P}^n), \quad (a,b) \mapsto ab$$

for $k+l \leq n$. This "intersection pairing" has the following property: if $X, Y \subset \mathbb{P}^n$ are two subvarieties that intersect in the expected dimension (i. e. $\operatorname{codim}(X \cap Y) = \operatorname{codim} X + \operatorname{codim} Y$) then $[X \cap Y] = [X] \cdot [Y]$. So "intersections of subvarieties can be performed on the level of cycle classes". As we have mentioned in the introduction to this section, the existence of such intersection pairing maps between the Chow groups will generalize to arbitrary smooth varieties. It is one of the most important properties of the Chow groups.

9.2. **Proper push-forward of cycles.**

We now want to generalize the push-forward maps of example 9.1.10 to more general morphisms, i. e. given a morphism $f : X \to Y$ of schemes we will study the question under which conditions there are induced push-forward maps $f_* : A_k(X) \to A_k(Y)$ for all k that are (roughly) given by $f_*[V] = [f(V)]$ for a k-dimensional subvariety V of X.

Remark 9.2.1. We have seen already in remark 9.1.12 (ii) that there are no such push-forward maps for the open inclusion $\mathbb{A}^1 \to \mathbb{P}^1$. The reason for this is precisely that the point $P = \mathbb{P}^1 \backslash \mathbb{A}^1$ is "missing" in the domain of the morphism: a rational function on $\mathbb{A}^1$ (which is then also a rational function on $\mathbb{P}^1$) may have a zero and / or pole at the point P which is then present on $\mathbb{P}^1$ but not on $\mathbb{A}^1$. As the class of P is not trivial in the Chow group of $\mathbb{P}^1$, this will change the rational equivalence class. Therefore there is no well-defined push-forward map between the Chow groups.

Another example of a morphism for which there is no push-forward for Chow groups is the trivial morphism $f : \mathbb{A}^1 \to \operatorname{pt}$: again the class of a point is trivial in $A_0(\mathbb{A}^1)$ but not in $A_0(\operatorname{pt})$. In contrast, the morphism $f : \mathbb{P}^1 \to \operatorname{pt}$ admits a well-defined push-forward map $f_* : A_0(\mathbb{P}^1) \cong \mathbb{Z} \to A_0(\operatorname{pt}) \cong \mathbb{Z}$ sending the class of a point in $\mathbb{P}^1$ to the class of a point in pt.

These counterexamples can be generalized by saying that in general there should be no points "missing" in the domain of the morphism $f : X \to Y$ for which we are looking for a push-forward f_*. For example, if Y is the one-pointed space, by "no points missing" we mean exactly that X should be compact (in the classical topology), i. e. complete in the sense of remark 3.4.5. For general Y we need a "relative version" of this compactness (resp. completeness) condition. Morphisms satisfying this condition are called *proper*. We will give both the topological definition (corresponding to "compactness") and the algebraic definition (corresponding to "completeness").

Definition 9.2.2. (Topological definition:) A continuous map $f : X \to Y$ of topological spaces is called **proper** if $f^{-1}(Z)$ is compact for every compact set $Z \subset Y$.

(Algebraic definition:) Let $f : X \to Y$ be a morphism of "nice" schemes (separated, of finite type over a field). For every morphism $g : Z \to Y$ from a third scheme Z form the fiber diagram

$$
\begin{array}{ccc}
X \times_Y Z & \longrightarrow & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{\ g\ } & Y.
\end{array}
$$

The morphism f is said to be **proper** if the induced morphism f' is closed for every such $g : Z \to Y$, i. e. if f' maps closed subsets of $X \times_Y Z$ to closed subsets of Z. This condition is sometimes expressed by saying that f is required to be "universally closed".

Remark 9.2.3. Note that the two definitions look quite different: whereas the topological definition places a condition on *inverse images* of (compact) subsets by some morphism, the algebraic definition places a condition on *images* of (closed) subsets by some morphism. Yet one can show that for varieties over the complex numbers the two definitions agree if we apply the topological definition to the classical (not the Zariski) topology. We will only illustrate this by some examples below. Note however that both definitions are "obvious" generalizations of their absolute versions, i. e. properness of a map in topology is a straightforward generalization of compactness of a space, whereas properness of a morphism in algebraic geometry is the expected generalization of completeness of a variety (see remark 3.4.5). In particular, if $Y = \text{pt}$ is a point then the (trivial) morphism $f : X \to \text{pt}$ is proper if and only if X is complete (resp. compact).

Example 9.2.4. If X is complete (resp. compact) then any morphism $f : X \to Y$ is proper. We will prove this both in the topological and the algebraic setting:

   (i) In topology, let $Z \subset Y$ be a compact subset of Y. In particular Z is closed, hence so is the inverse image $f^{-1}(Z)$ as f is continuous. It follows that $f^{-1}(Z)$ is a closed subset of a compact space X, hence compact.

   (ii) In algebra, the fiber product $X \times_Y Z$ in definition 9.2.2 is isomorphic to the closed subscheme $p^{-1}(\Delta_Y)$ of $X \times Z$, where $p = (f,g) : X \times Z \to Y \times Y$ and $\Delta_Y \subset Y \times Y$ is the diagonal. So if $V \subset X \times_Y Z$ is any closed subset, then V is also closed in $X \times Z$, and hence its image in Z is closed as X is complete.

This is the easiest criterion to determine that a morphism is proper. Some more can be found in exercise 9.5.5.

Example 9.2.5. Let $U \subset X$ be a non-empty open subset of a (connected) scheme X. Then the inclusion morphism $i : U \to X$ is not proper. This is obvious for the algebraic definition, as i is not even closed itself (it maps the closed subset $U \subset U$ to the non-closed subset $U \subset X$). In the topological definition, let $Z \subset X$ be a small closed disc around a point $P \in X \backslash U$. Its inverse image $i^{-1}(Z) = Z \cap U$ is Z minus a closed non-empty subset, so it is not compact.

Example 9.2.6. If $f : X \to Y$ is proper then every fiber $f^{-1}(P)$ is complete (resp. compact). Again this is obvious for the topological definition, as $\{P\} \subset Y$ is compact. In the algebraic definition let $P \in Y$ be a point, let Z be any scheme, and form the fiber diagram

$$
\begin{array}{ccccc}
Z \times f^{-1}(P) & \longrightarrow & f^{-1}(P) & \longrightarrow & X \\
\downarrow{\scriptstyle f'} & & \downarrow & & \downarrow{\scriptstyle f} \\
Z & \longrightarrow & P & \longrightarrow & Y.
\end{array}
$$

If f is proper then by definition the morphism f' is closed for all choices of P and Z. By definition this means exactly that all fibers $f^{-1}(P)$ of f are complete.

The converse is not true however: every fiber of the morphism $\mathbb{A}^1 \to \mathbb{P}^1$ is complete (resp. compact), but the morphism is not proper.

Remark 9.2.7. It turns out that the condition of properness of a morphism $f : X \to Y$ is enough to guarantee the existence of well-defined push-forward maps $f_* : A_k(X) \to A_k(Y)$. To construct them rigorously however we have to elaborate further on our idea that f_* should map any k-dimensional cycle $[V]$ to $[f(V)]$, as the following two complications can occur:

(i) The image $f(V)$ of V may have dimension smaller than k, so that $f(V)$ does not define a k-dimensional cycle. It turns out that we can consistently define $f_*[V]$ to be zero in this case.

(ii) It may happen that $\dim f(V) = \dim V$ and the morphism f is a *multiple covering map*, i.e. that a general point in $f(V)$ has $d > 1$ inverse image points. In this case the image $f(V)$ is "covered d times by V", so we would expect that we have to set $f_*[V] = d \cdot [f(V)]$. Let us define this "order of the covering" d rigorously:

Proposition 9.2.8. *Let $f : X \to Y$ be a morphism of varieties of the same dimension such that $f(X)$ is dense in Y. Then:*

(i) *$K(X)$ is a finite-dimensional vector space over $K(Y)$. Its dimension is called the **degree** of the morphism f, denoted $\deg f$. (One also says that $K(X) : K(Y)$ is a **field extension** of dimension $[K(X) : K(Y)] = \deg f$.)*

(ii) *The degree of f is equal to the number of points in a general fiber of f. (This means: there is a non-empty open set $U \subset Y$ such that the fibers of f over U consist of exactly $\deg f$ points.)*

(iii) *If moreover f is proper then every zero-dimensional fiber of f consists of exactly $\deg f$ points if the points are counted with their scheme-theoretic multiplicities.*

Proof. (i): We begin with a few reduction steps. As the fields of rational functions do not change when we pass to an open subset, we can assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine. Next, we factor the morphism $f : X \to Y$ as $f = \pi \circ \gamma$ with $\gamma : X \to \Gamma \subset X \times Y$ the graph morphism and $\pi : X \times Y \to Y$ the projection. As γ is an isomorphism it is sufficient to show the statement of the proposition for the projection map π. Finally, we can factor the projection π (which is the restriction of the obvious projection map $\mathbb{A}^{n+m} \to \mathbb{A}^m$ to $X \times Y$) into n projections that are given by dropping one coordinate at a time. Hence we can assume that $X \subset \mathbb{A}^{n+1}$ and $Y \subset \mathbb{A}^n$, and prove the statement for the map $\pi : X \to Y$ that is the restriction of the projection map $(x_0, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$ to X.

In this case the field $K(X)$ is generated over $K(Y)$ by the single element x_0. Assume that $x_0 \in K(X)$ is transcendental over $K(Y)$, i.e. there is no polynomial relation of the form

$$F_d x_0^d + F_{d-1} x_0^{d-1} \cdots + F_0 = 0, \tag{$*$}$$

for $F_i \in K(Y)$ and $F_d \neq 0$. Then for every choice of $(x_1, \ldots, x_n) \in Y$ the value of x_0 in X is not restricted, i.e. the general fiber of f is not finite. But then $\dim X > \dim Y$ in contradiction to our assumption. So $x_0 \in K(X)$ is algebraic over $K(Y)$, i.e. there is a relation $(*)$. It follows that $K(X)$ is a vector space over $K(Y)$ with basis $\{1, x_0, \ldots, x_0^{d-1}\}$.

(ii): Continuing the proof of (i), note that on the non-empty open subset of Y where all F_i are regular and F_d is non-zero every point in the target has exactly d inverse image points (counted with multiplicity). Restricting the open subset further to the open subset where the discriminant of the polynomial $(*)$ is non-zero, we can in fact show that there is an open subset of Y on which the inverse images of f consist set-theoretically of exactly d points that all count with multiplicity 1.

(iii): We will only sketch this part, using the topological definition of properness. By (ii) there is an open subset $U \subset Y$ on which all fibers of f consist of exactly n points. Let $P \in Y$ be any point, and choose a small closed disc $\Delta \subset U \cup \{P\}$ around P. If Δ is small enough then the inverse image $f^{-1}(\Delta \backslash \{P\})$ will be a union of d copies of $\Delta \backslash \{P\}$. As f is proper, the inverse image $f^{-1}(\Delta)$ has to be compact, i.e. all the holes in the d copies of $\Delta \backslash \{P\}$ have to be filled in by inverse image points of P. So the fiber $f^{-1}(P)$ must contain at least d points (counted with multiplicities). But we see from $(*)$ above that every fiber contains at most d points unless it is infinite (i.e. all F_i are zero at P). This shows part (iii). $\qquad\square$

We are now ready to construct the push-forward maps $f_* : A_k(X) \to A_k(Y)$ for proper morphisms $f : X \to Y$.

Construction 9.2.9. Let $f : X \to Y$ be a proper morphism of schemes. Then for every subvariety $Z \subset X$ the image $f(Z)$ is a closed subvariety of dimension at most $\dim Z$. On the cycle level we define homomorphisms $f_* : Z_k(X) \to Z_k(Y)$ by

$$f_*[Z] = \begin{cases} [K(Z) : K(f(Z))] \cdot [f(Z)] & \text{if } \dim f(Z) = \dim Z, \\ 0 & \text{if } \dim f(Z) < \dim Z. \end{cases}$$

By proposition 9.2.8 this is well-defined and corresponds to the ideas mentioned in remark 9.2.7.

Remark 9.2.10. By the multiplicativity of degrees of field extensions it follows that the push-forwards are functorial, i. e. $(g \circ f)_* = g_* f_*$ for any two morphisms $f : X \to Y$ and $g : Y \to Z$.

Of course we want to show that these homomorphisms pass to the Chow groups, i. e. give rise to well-defined homomorphisms $f_* : A_k(X) \to A_k(Y)$. For this we have to show by definition that divisors of rational functions are pushed forward to divisors of rational functions.

Theorem 9.2.11. *Let $f : X \to Y$ be a proper surjective morphism of varieties, and let $\varphi \in K(X)^*$ be a non-zero rational function on X. Then*

$$f_* \operatorname{div}(\varphi) = \begin{cases} 0 & \text{if } \dim Y < \dim X \\ \operatorname{div}(N(\varphi)) & \text{if } \dim Y = \dim X \end{cases}$$

*in $Z_*(Y)$, where $N(\varphi) \in K(Y)$ denotes the determinant of the endomorphism of the $K(Y)$-vector space $K(X)$ given by multiplication by φ (this is usually called the **norm** of φ).*

Proof. The complete proof of the theorem with all algebraic details is beyond the scope of these notes; it can be found in [F] proposition 1.4. We will only sketch the idea of the proof here.

Case 1: $\dim Y < \dim X$ (see the picture below). We can assume that $\dim Y = \dim X - 1$, as otherwise the statement is trivial for dimensional reasons. Note that we must have $f_* \operatorname{div}(\varphi) = n \cdot [Y]$ for some $n \in \mathbb{Z}$ by example 9.1.5. So it only remains to determine the number n. By our interpretation of remark 9.2.7 (ii) we can compute this number on a general fiber of f by counting all points in this fiber with the multiplicity with which they occur in the restriction of φ to this fiber. In other words, we have $n = \sum_{P:f(P)=Q} \operatorname{ord}_P(\varphi|_{f^{-1}(Q)})$ for any point $Q \in Y$ over which the fiber of f is finite. But this number is precisely the degree of $\varphi|_{f^{-1}(Q)}$ on the complete curve $f^{-1}(Q)$, which must be zero. (Strictly speaking we have only shown this for smooth projective curves in remark 6.3.5, but it is true in the general case as well. The important ingredient is here that the fiber is complete.)

Case 2: $\dim Y = \dim X$ (see the picture below). We will restrict ourselves here to showing the stated equation *set-theoretically*, i. e. we will assume that φ is (locally around a fiber) a regular function and show that $f(Z(\varphi)) = Z(N(\varphi))$, where $Z(\cdot)$ denotes as usual the zero locus of a function.

Note first that we can neglect the fibers of f that are not finite: these fibers can only lie over a subset of Y of codimension at least 2 (otherwise the non-zero-dimensional fibers would form a component of X for dimensional reasons, in contrast to X being irreducible). So as $f_* \operatorname{div}(\varphi)$ is a cycle of codimension 1 in Y these higher-dimensional fibers cannot contribute to the push-forward.

Case 1   Case 2
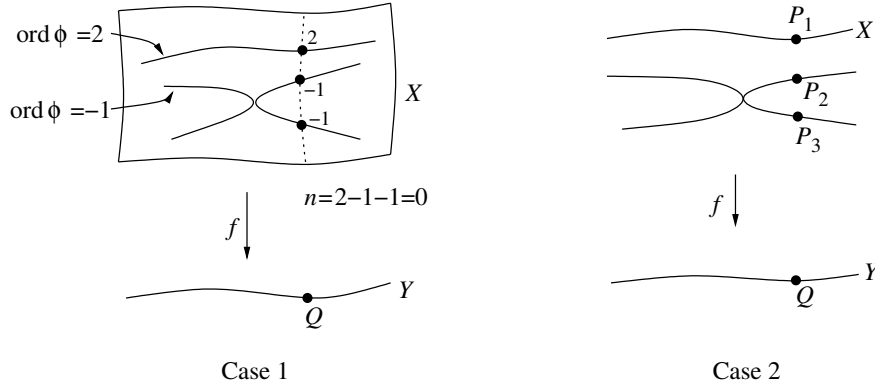
Now let $Q \in Y$ be any point such that the fiber $f^{-1}(Q)$ is finite. Then $f^{-1}(Q)$ consists of exactly $d = [K(X):K(Y)]$ points by proposition 9.2.8 (iii). Let us assume for simplicity that all these points are distinct (although this is not essential), so $f^{-1}(Q) = \{P_1,\ldots,P_d\}$. The space of functions on this fiber is then just k^d, corresponding to the value at the d points. In this basis, the restriction of the function φ to this fiber is then obviously given by the diagonal matrix with entries $\varphi(P_1),\ldots,\varphi(P_d)$, so its determinant is $N(\varphi)(Q) = \prod_{i=1}^d \varphi(P_i)$. Now it is clear that

$$Q \in f(Z(\varphi)) \iff \text{there is a } P_i \text{ over } Q \text{ with } \varphi(P_i) = 0$$
$$\iff Q \in Z(N(\varphi)).$$

We can actually see the multiplicities arising as well: if there are k points among the P_i where φ vanishes, then the diagonal matrix $\varphi|_{f^{-1}(Q)}$ contains k zeros on the diagonal, hence its determinant is a product that contains k zeros, so it should give rise to a zero of order k, in accordance with our interpretation of remark 9.2.7 (ii).   □

Corollary 9.2.12. *Let $f : X \to Y$ be a proper morphism of schemes. Then there are well-defined push-forward maps $f_* : A_k(X) \to A_k(Y)$ for all $k \geq 0$ given by the definition of construction 9.2.9.*

Proof. This follows immediately from theorem 9.2.11 applied to the morphism from a $(k+1)$-dimensional subvariety of X to its image in Y.   □

Example 9.2.13. Let X be a complete scheme, and let $f : X \to \text{pt}$ be the natural (proper) map. For any 0-dimensional cycle class $\alpha \in A_0(X)$ we define the **degree** of α to be the integer $f_*\alpha \in A_0(\text{pt}) \cong \mathbb{Z}$. This is well-defined by corollary 9.2.12. More explicitly, if $\alpha = \sum_i n_i[P_i]$ for some points $P_i \in X$ then $\deg \alpha = \sum_i n_i$.

Example 9.2.14. Let $X = \tilde{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ with coordinates $(x_0 : x_1 : x_2)$ in the point $P = (1:0:0)$, and denote by $E \subset X$ the exceptional hypersurface. In this example we will compute the Chow groups of X using remark 9.1.18.

Note that $\mathbb{P}^2$ has a stratification by affine spaces as $\mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^0$. Identifying $\mathbb{A}^0$ with P and recalling that the blow-up $\tilde{\mathbb{P}}^2$ is obtained from $\mathbb{P}^2$ by "replacing the point P with a line $\mathbb{P}^1$" we see that X has a stratification $\mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^1 \cup \mathbb{A}^0$. By remark 9.1.18 it follows that the closures of these four strata generate $A_*(X)$. More precisely, these four classes are $[X] \in A_2(X)$, $[L] \in A_1(X)$ where L is the strict transform of a line in $\mathbb{P}^2$ through P, the exceptional hypersurface $[E] \in A_1(X)$, and the class of a point in $A_0(X)$. It follows immediately that $A_2(X) \cong \mathbb{Z}$ and $A_0(X) \cong \mathbb{Z}$. Moreover we see that $A_1(X)$ is generated by $[L]$ and $[E]$.

We have already stated without proof in remark 9.1.18 that $[L]$ and $[E]$ form in fact a basis of $A_1(X)$. Let us now prove this in our special case at hand. So assume that there is a relation $n[L] + m[E] = 0$ in $A_1(X)$. Consider the following two morphisms:

(i) Let $\pi : X \to \mathbb{P}^2$ be the projection to the base of the blow-up. This is a proper map, and we have $\pi_*[L] = [H]$ and $\pi_*[E] = 0$ where $[H] \in A_1(\mathbb{P}^2)$ is the class of a line. So we see that

$$0 = \pi_*(0) = \pi_*(n[L] + m[E]) = n[H] \in A_1(\mathbb{P}^2),$$

from which we conclude that $n = 0$.

(ii) Now let $p : X \to \mathbb{P}^1$ be the morphism that is the identity on E, and sends every point $Q \in X \backslash E$ to the unique intersection point of E with the strict transform of the line through P and Q. Again this is a proper map, and we have $p_*[L] = 0$ and $p_*[E] = [\mathbb{P}^1]$. So again we see that

$$0 = p_*(0) = p_*(n[L] + m[E]) = m[\mathbb{P}^1] \in A_1(\mathbb{P}^1),$$

from which we conclude that $m = 0$ as well.

Combining both parts we see that there is no non-trivial relation of the form $n[L] + m[E] = 0$ in $A_1(X)$.

Now let $[H]$ be the class of a line in X that does not intersect the exceptional hypersurface. We have just shown that $[H]$ must be a linear combination of $[L]$ and $[E]$. To compute which one it is, consider the rational function $\frac{x_1}{x_0}$ on X. It has simple zeros along L and E, and a simple pole along H (with coordinates for L and H chosen appropriately). So we conclude that $[H] = [L] + [E]$ in $A_1(X)$.

9.3. **Weil and Cartier divisors.** Our next goal is to describe intersections on the level of Chow groups as motivated in the beginning of section 9.1. We will start with the easiest case, namely with the intersection of a variety with a subset of codimension 1. To put it more precisely, given a subvariety $V \subset X$ of dimension k and another one $D \subset X$ of codimension 1, we want to construct an *intersection cycle* $[V] \cdot [D] \in A_{k-1}(X)$ with the property that $[V] \cdot [D] = [V \cap D]$ if this intersection $V \cap D$ actually has dimension $k - 1$. Of course these intersection cycles should be well-defined on the Chow groups, i. e. the product cycle $[V] \cdot [D] \in A_{k-1}(X)$ should only depend on the classes of V and D in $A_*(X)$.

Example 9.3.1. Here is an example showing that this is too much to hope for in the generality as we stated it. Let $X = \mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$ be the union of two projective planes glued together along a common line. Let $L_1, L_2, L_3 \subset X$ be the lines as in the following picture.
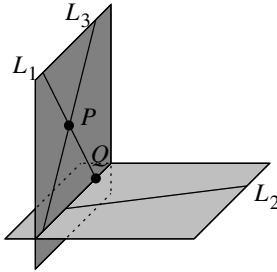
Their classes in $A_1(X)$ are all the same since $A_1(X) \cong \mathbb{Z}$ by remark 9.1.18. But note that $L_1 \cap L_2$ is empty, whereas $L_1 \cap L_3$ is a single point P. But $0 \neq [P] \in A_0(X)$, so there can be no well-defined product map $A_1(X) \times A_1(X) \to A_0(X)$ that describes intersections on this space X.

The reason why this construction failed is quite a subtle one: we have to distinguish between codimension-1 subspaces and spaces that can locally be written as the zero locus of a single function. In general the intersection product exists only for intersections with spaces that are locally the zero locus of a single function. For most spaces this is the same thing as codimension-1 subspaces, but notably not in example 9.3.1 above: neither of the three lines L_i can be written as the zero locus of a single function *on* X: there is a

(linear) function on the vertical $\mathbb{P}^2$ that vanishes precisely on L_1, but we cannot extend it to a function on all of X that vanishes at the point Q but nowhere else on the horizontal $\mathbb{P}^2$. (We can write the L_i as the zero locus of a single function *on a component of X*, but this is not what we need.)

So for intersection-theoretic purposes we have to make a clear distinction between codimension-1 subspaces and spaces that are locally the zero locus of a single function. Let us make the corresponding definitions.

Definition 9.3.2. Let X be a scheme.

  (i) If X has pure dimension n a **Weil divisor** on X is an element of $Z_{n-1}(X)$. Obviously, the Weil divisors form an Abelian group. Two Weil divisors are called **linearly equivalent** if they define the same class in $A_{n-1}(X)$. The quotient group $A_{n-1}(X)$ is called the group of **Weil divisor classes**.

  (ii) Let $\mathcal{K}_X$ be the sheaf of rational functions on X, and denote by $\mathcal{K}_X^*$ the subsheaf of invertible elements (i. e. of those functions that are not identically zero on any component of X). Note that $\mathcal{K}_X^*$ is a sheaf of Abelian groups, with the group structure given by multiplication of rational functions. Similarly, let O_X^* be the sheaf of invertible elements of O_X (i. e. of the regular functions that are nowhere zero). Note that O_X^* is a sheaf of Abelian groups under multiplication as well. In fact, O_X^* is a subsheaf of $\mathcal{K}_X^*$.

  A **Cartier divisor** on X is a global section of the sheaf $\mathcal{K}_X^*/O_X^*$. Obviously, the Cartier divisors form an Abelian group under multiplication, denoted $\operatorname{Div} X$. In analogy to Weil divisors the group structure on $\operatorname{Div} X$ is usually written additively however. A Cartier divisor is called linearly equivalent to zero if it is induced by a global section of $\mathcal{K}_X^*$. Two Cartier divisors are **linearly equivalent** if their difference (i. e. quotient, see above) is linearly equivalent to zero. The quotient group $\operatorname{Pic} X := \Gamma(\mathcal{K}_X^*/O_X^*)/\Gamma(\mathcal{K}_X^*)$ is called the group of **Cartier divisor classes**.

Remark 9.3.3. Let us analyze the definition of Cartier divisors. There is an obvious exact sequence of sheaves on X

$$0 \to O_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/O_X^* \to 0.$$

Note that these are *not* sheaves of O_X-modules, so their flavor is slightly different from the ones we have considered so far. But it is still true that we get an exact sequence of global sections

$$0 \to \Gamma(O_X^*) \to \Gamma(\mathcal{K}_X^*) \to \Gamma(\mathcal{K}_X^*/O_X^*)$$

that is in general not exact on the right. More precisely, recall that the quotient sheaf $\mathcal{K}_X^*/O_X^*$ is *not* just the sheaf that is $\mathcal{K}_X^*(U)/O_X^*(U)$ for all open subsets $U \subset X$, but rather the sheaf associated to this presheaf. Therefore $\Gamma(\mathcal{K}_X^*/O_X^*)$ is in general not just the quotient $\Gamma(\mathcal{K}_X^*)/\Gamma(O_X^*)$.

To unwind the definition of sheafification, an element of $\operatorname{Div} X = \Gamma(\mathcal{K}_X^*/O_X^*)$ can be given by a (sufficiently fine) open covering $\{U_i\}$ and elements of $\mathcal{K}_X^*(U_i)/O_X^*(U_i)$ represented by rational functions φ_i for all i such that their quotients $\frac{\varphi_i}{\varphi_j}$ are in $O_X^*(U_i \cap U_j)$ for all i, j. So a Cartier divisor is an object that is locally a (non-zero) rational function modulo a nowhere-zero regular function. Intuitively speaking, the only data left from a rational function if we mod out locally by nowhere-zero regular functions is the locus of its zeros and poles together with their multiplicities. So one can think of Cartier divisors as objects that are (linear combinations of) zero loci of functions.

A Cartier divisor is linearly equivalent to zero if it is *globally* a rational function, just the same as for Weil divisors. From cohomology one would expect that one can think of the quotient group $\operatorname{Pic} X$ as the cohomology group $H^1(X, O_X^*)$. We cannot say this rigorously because we have only defined cohomology for quasi-coherent sheaves (which O_X^* is not).

But there is a more general theory of cohomology of arbitrary sheaves of Abelian groups on schemes, and in this theory the statement that $\text{Pic} X = H^1(X, O_X^*)$ is correct.

Lemma 9.3.4. *Let X be a purely n-dimensional scheme. Then there is a natural homomorphism* $\text{Div} X \to Z_{n-1}(X)$ *that passes to linear equivalence to give a homomorphism* $\text{Pic} X \to A_{n-1}(X)$. *In other words, every Cartier divisor (class) determines a Weil divisor (class).*

Proof. Let $D \in \text{Div} X$ be a Cartier divisor on X, represented by an open covering $\{U_i\}$ of X and rational functions φ_i on U_i. For any $(n-1)$-dimensional subvariety V of X define the order of D at V to be $\text{ord}_V D := \text{ord}_{V \cap U_i} \varphi_i$, where i is an index such that $U_i \cap V \neq \emptyset$. This does not depend on the choice of i as the quotients $\frac{\varphi_i}{\varphi_j}$ are nowhere-zero regular functions, so the orders of φ_i and φ_j are the same where they are both defined. So we get a well-defined map $\text{Div} X \to Z_{n-1}(X)$ defined by $D \mapsto \sum_V \text{ord}_V D \cdot [V]$. It is obviously a homomorphism as $\text{ord}_V(\varphi_i \cdot \varphi_i') = \text{ord}_V \varphi_i + \text{ord}_V \varphi_i'$.

It is clear from the definition that a Cartier divisor that is linearly equivalent to zero, i.e. a global rational function, determines a Weil divisor in $B_{n-1}(X)$. Hence the homomorphism passes to linear equivalence. $\square$

Lemma 9.3.5. *Let X be a smooth projective curve. Then Cartier divisors (resp. Cartier divisor classes) on X are the same as Weil divisors (resp. Weil divisor classes). In particular, our definition 9.3.2 (ii) of* $\text{Div} X$ *and* $\text{Pic} X$ *agrees with our earlier one from section 6.3.*

Proof. The idea of the proof is lemma 7.5.6 which tells us that every point of X is locally the scheme-theoretic zero locus of a single function, hence a Cartier divisor.

To be more precise, let $\sum_{i=1}^n a_i P_i \in Z_0(X)$ be a Weil divisor. We will construct a Cartier divisor $D \in \text{Div} X$ that maps to the given Weil divisor under the correspondence of lemma 9.3.4. To do so, pick an open neighborhood U_i of P_i for all $i = 1, \ldots, n$ such that

(i) $P_j \notin U_i$ for $j \neq i$, and
(ii) there is a function φ_{P_i} on U_i such that $\text{div} \varphi_{P_i} = 1 \cdot P_i$ on U_i (see lemma 7.5.6).

Moreover, set $U = X \backslash \{P_1, \ldots, P_n\}$. Then we define a Cartier divisor D by the open cover $\{U, U_1, \ldots, U_n\}$ and the rational functions

(i) 1 on U,
(ii) $\varphi_{P_i}^{a_i}$ on U_i.

Note that these data define a Cartier divisor: no intersection of two elements of the open cover contains one of the points P_i, and the functions given on the elements of the open cover are regular and non-vanishing away from the P_i. By construction, the Weil divisor associated to D is precisely $\sum_{i=1}^n a_i P_i$, as desired. $\square$

Example 9.3.6. In general, the map from Cartier divisors (resp. Cartier divisor classes) to Weil divisors (resp. Cartier divisor classes) is neither injective nor surjective. Here are examples of this:

(i) not injective: This is essentially example 9.1.7. Let $X = X_1 \cup X_2$ be the union of two lines $X_i \cong \mathbb{P}^1$ glued together at a point $P \in X_1 \cap X_2$. Let Q be a point on $X_1 \backslash X_2$. Consider the open cover $X = U \cup V$ with $U = X \backslash Q$ and $V = X_1 \backslash P$.

We define a Cartier divisor D on X by choosing the following rational functions on U and V: the constant function 1 on U, and the linear function on $V \cong \mathbb{A}^1$ that has a simple zero at Q. Note that the quotient of these two functions is regular and nowhere zero on $U \cap V$, so D is well-defined. Its associated Weil divisor $[D]$ is $[Q]$.

By symmetry, we can construct a similar Cartier divisor D' whose associated Weil divisor is the class of a point $Q' \in X_2 \backslash X_1$.

Now note that the Cartier divisor classes of D and D' are different (because $D - D'$ is not the divisor of a rational function), but their associated Weil divisors $[Q]$ and $[Q']$ are the same by example 9.1.7.

(ii) not surjective: This is essentially example 9.3.1. The classes $[L_i]$ of this example are Weil divisors but not Cartier divisors.

Another example on an irreducible space X is the cone

$$X = \{x_3^2 = x_1^2 + x_2^2\} \subset \mathbb{P}^3.$$
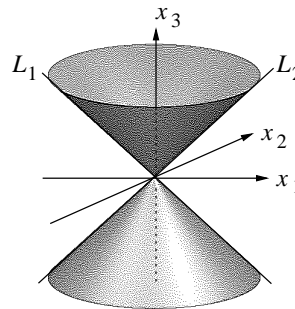
Let $L_1 = Z(x_2, x_1 + x_3)$ and $L_2 = Z(x_2, x_1 - x_3)$ be the two lines as in the picture. We claim that there is no Cartier divisor on X corresponding to the Weil divisor $[L_1]$. In fact, if there was such a Cartier divisor, defined locally around the origin by a function φ, we must have an equality of ideals

$$(x_1^2 + x_2^2 - x_3^2, \varphi) = (x_2, x_1 + x_3)$$

in the local ring $O_{\mathbb{P}^3, 0}$. This is impossible since the right ideal contains two linearly independent linear parts, whereas the left ideal contains only one. But note that the section x_2 of the line bundle $O_X(1)$ defines a Cartier divisor $\mathrm{div}(x_2)$ on X whose associated Weil divisor is $[L_1] + [L_2]$, and the section $x_1 + x_3$ defines a Cartier divisor whose associated Weil divisor is $2[L_1]$. So $[L_1]$ and $[L_2]$ are not Cartier divisors, whereas $[L_1] + [L_2]$, $2[L_1]$, and $2[L_2]$ are. In particular, there is in general no "decomposition of a Cartier divisor into its irreducible components" as we have it by definition for Weil divisors.

There is quite a deep theorem however that the two notions agree on smooth schemes:

Theorem 9.3.7. *Let X be a smooth n-dimensional scheme. Then $\mathrm{Div}\, X \cong Z_{n-1}(X)$ and $\mathrm{Pic}\, X \cong A_{n-1}(X)$.*

Proof. We cannot prove this here and refer to [H] remark II.6.11.1.A for details. One has to prove the analogue of lemma 7.5.6, i.e. that every codimension-1 subvariety of X is locally the scheme-theoretic zero locus of a single function. This is a commutative algebra statement as it can be shown on the local ring of X at the subvariety.

(To be a little more precise, the property of X that we need is that its local rings are unique factorization domains: if this is the case and $V \subset X$ is an subvariety of codimension 1, pick any non-zero (local) function $f \in O_{X,V}$ that vanishes on V. As $O_{X,V}$ is a unique factorization domain we can decompose f into its irreducible factors $f = f_1 \cdots f_n$. Of course one of the f_i has to vanish on V. But as f_i is irreducible, its ideal must be the ideal of V, so V is locally the zero locus of a single function. The problem with this is that it is almost impossible to check that a ring (that one does not know very well) is a unique factorization domain. So one uses the result from commutative algebra that every *regular* local ring (i.e. "the local ring of a scheme at a *smooth* point") is a unique factorization

domain. Actually, we can see from the above argument that it is enough that X is "smooth in codimension 1", i.e. that its set of singular points has codimension at least 2 — or to express it algebraically, that its local rings $O_{X,V}$ at codimension-1 subvarieties V are regular.)   □

Example 9.3.8. Finally let us discuss the relation between divisors and line bundles as observed for curves in section 7.5. Note that we have in fact used such a correspondence already in example 9.3.6 where we defined a Cartier divisor by giving a section of a line bundle. The precise relation between line bundles and Cartier divisors is as follows.

Lemma 9.3.9. *For any scheme X there are one-to-one correspondences*

$$\{Cartier\ divisors\ on\ X\} \leftrightarrow \{(\mathcal{L},s)\ ;\ \mathcal{L}\ a\ line\ bundle\ on\ X\ and\ s\ a\ rational\ section\ of\ \mathcal{L}\}$$

and

$$\{Cartier\ divisor\ classes\ on\ X\} \leftrightarrow \{line\ bundles\ on\ X\ that\ admit\ a\ rational\ section\}.$$

Proof. The proof of this is essentially the same as the correspondence between divisor classes and line bundles on a smooth projective curve in proposition 7.5.9. Given a Cartier divisor $D = \{(U_i, \varphi_i)\}$ on X, we get an associated line bundle $O(D)$ by taking the subsheaf of O_X-modules of $\mathcal{K}_X$ generated by the functions $\frac{1}{\varphi_i}$ on U_i. Conversely, given a line bundle with a rational section, this section immediately defines a Cartier divisor. The proof that the same correspondence holds for divisor classes is the same as in proposition 7.5.9.   □

Remark 9.3.10. We should note that almost any line bundle on any scheme X admits a rational section. In fact, this is certainly true for irreducible X (as the line bundle is then isomorphic to the structure sheaf on a dense open subset of X by definition), and one can show that it is true in most other cases as well (see [H] remark 6.14.1 for more information). Most books actually *define* the group $\operatorname{Pic} X$ to be the group of line bundles on X.

Summarizing our above discussions we get the following commutative diagram:

where

    (i) the bottom row (the Weil divisors) exists only if X is purely n-dimensional,
   (ii) the upper right vertical arrow is an isomorphism in most cases, at least if X is irreducible,
  (iii) the lower vertical arrows are isomorphisms at least if X is smooth (in codimension 1).

Remark 9.3.11. Although line bundles, Cartier divisor classes, and Weil divisor classes are very much related and even all the same thing on many schemes (e.g. smooth varieties), note that their "functorial properties" are quite different: if $f : X \to Y$ is a morphism then for line bundles and Cartier divisors the pull-back f^* is the natural operation, whereas for Weil divisors (i.e. elements of the Chow groups) the push-forward f_* as in section 9.2 is more natural. In algebraic topology this can be expressed by saying that Weil divisors correspond to *homology* cycles, whereas Cartier divisors correspond to *cohomology* cycles. On nice spaces this is the same by Poincaré duality, but this is a non-trivial statement. The

natural operation for homology (resp. cohomology) is the push-forward (resp. pull-back). Intersection products are defined between a cohomology and a homology class, yielding a homology class. This corresponds to our initial statement of this section that intersection products of Chow cycles ("homology classes") with divisors will usually only be well-defined with Cartier divisors ("cohomology classes") and not with Weil divisors.

9.4. **Intersections with Cartier divisors.** We are now ready to define intersection products of Chow cycles with Cartier divisors, as motivated in the beginning of section 9.3. Let us give the definition first, and then discuss some of its features.

Definition 9.4.1. Let X be a scheme, let $V \subset X$ be a k-dimensional subvariety with inclusion morphism $i : V \to X$, and let D be a Cartier divisor on X. We define the **intersection product** $D \cdot V \in A_{k-1}(X)$ to be

$$D \cdot V = i_*[i^* O_X(D)],$$

where $O_X(D)$ is the line bundle on X associated to the Cartier divisor D by lemma 9.3.9, i^* denotes the pull-back of line bundles, $[i^* O_X(D)]$ is the Weil divisor class associated to the line bundle $i^* O_X(D)$ by remark 9.3.10 (note that V is irreducible), and i_* denotes the proper push-forward of corollary 9.2.12.

Note that by definition the intersection product depends only on the divisor class of D, not on D itself. So using our definition we can construct bilinear intersection products

$$\operatorname{Pic} X \times Z_k(X) \to A_{k-1}(X), \quad \left(D, \sum a_i [V_i]\right) \mapsto \sum a_i (D \cdot V_i).$$

If X is smooth and pure-dimensional (so that Weil and Cartier divisors agree) and W is a codimension-1 subvariety of X, we denote by $W \cdot V \in A_{k-1}(X)$ the intersection product $D \cdot V$, where D is the Cartier divisor corresponding to the Weil divisor $[W]$.

Example 9.4.2. Let X be a smooth n-dimensional scheme, and let V and W be subvarieties of dimensions k and $n-1$, respectively. If $V \not\subset W$, i.e. if $\dim(W \cap V) = k-1$, then the intersection product $W \cdot V$ is just the cycle $[W \cap V]$ with possibly some scheme-theoretic multiplicities. In fact, in this case the Weil divisor $[W]$ corresponds by remark 9.3.10 to a line bundle $O_X(W)$ together with a section f whose zero locus is precisely W. By definition of the intersection product we have to pull back this line bundle to V, i.e. restrict the section f to V. The cycle $W \cdot V$ is then the zero locus of $f|_V$, with possibly scheme-theoretic multiplicities if f vanishes along V with higher order.

As a concrete example, let C_1 and C_2 be two curves in $\mathbb{P}^2$ of degrees d_1 and d_2, respectively, that intersect in finitely many points $P_1, \ldots, P_n$. Then the intersection product $C_1 \cdot C_2 \in A_0(\mathbb{P}^2)$ is just $\sum_i a_i [P_i]$, where a_i is the scheme-theoretic multiplicity of the point P_i in the intersection scheme $C_1 \cap C_2$. Using that all points in $\mathbb{P}^2$ are rationally equivalent, i.e. that $A_0(\mathbb{P}^2) \cong \mathbb{Z}$ is generated by the class of any point, we see that $C_1 \cdot C_2$ is just the Bézout number $d_1 \cdot d_2$.

Example 9.4.3. Again let X be a smooth n-dimensional scheme, and let V and W be subvarieties of dimensions k and $n-1$, respectively. This time let us assume that $V \subset W$, so that the intersection $W \cap V = V$ has dimension k and thus does not define a $(k-1)$-dimensional cycle. There are two ways to interpret the intersection product $W \cdot V$ in this case:

(i) Recall that the intersection product $W \cdot V$ depends only on the divisor *class* of W, not on W itself. So if we can replace W by a linearly equivalent divisor W' such that $V \not\subset W'$ then the intersection product $W \cdot V$ is just $W' \cdot V$ which can now be constructed as in example 9.4.2. For example, let $H \subset \mathbb{P}^2$ be a line and assume that we want to compute the intersection product $H \cdot H \in A_0(\mathbb{P}^2) \cong \mathbb{Z}$. The intersection $H \cap H$ has dimension 1, but we can move the first H to a different line H' which is linearly equivalent to H. So we see that $H \cdot H = H' \cdot H = 1$, as

$H' \cap H$ is just one point. Note however that it may not always be possible to find such a linearly equivalent divisor that makes the intersection have the expected dimension.

(ii) If the strategy of (i) does not work or one does not want to apply it, there is also a different description of the intersection product for which no moving of W is necessary. Let us assume for simplicity that W is smooth. By the analogue of remark 7.4.17 for general hypersurfaces the bundle $i^* O_X(W)$ (where $i : V \to X$ is the inclusion morphism) is precisely the restriction to V of the normal bundle $N_{W/X}$ of W in X. By definition 9.4.1 the intersection product $W \cdot V$ is then the Weil divisor associated to this bundle, i. e. the locus of zeros minus poles of a rational section of the normal bundle $N_{W/X}$ restricted to V.

$$W \cdot V = [P_1] + [P_2]$$
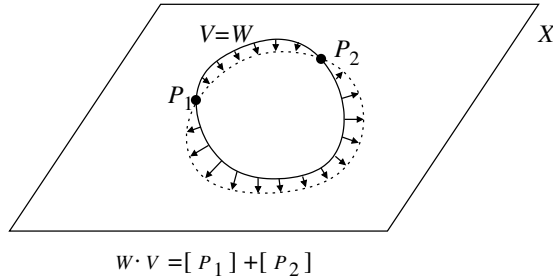
Note that we can consider this procedure as an infinitesimal version of (i): the section of the normal bundle describes an "infinitesimal deformation" of W in X, and the deformed W meets V precisely in the locus where the section vanishes.

Proposition 9.4.4. *(Commutativity of the intersection product) Let X be an n-dimensional variety, and let D_1, D_2 be Cartier divisors on X with associated Weil divisors $[D_1], [D_2]$. Then $D_1 \cdot [D_2] = D_2 \cdot [D_1] \in A_{n-2}(X)$.*

Proof. We will only sketch the proof in two easy cases (that cover most applications however). For the general proof we refer to [F] theorem 2.4.

Case 1: D_1 and D_2 intersect in the expected dimension, i. e. the locus where the defining equations of both D_1 and D_2 have a zero or pole has codimension 2 in X. Then one can show that both $D_1 \cdot [D_2]$ and $D_2 \cdot [D_1]$ is simply the sum of the components of the geometric intersection $D_1 \cap D_2$, counted with their scheme-theoretic multiplicities. In other words, if $V \subset X$ is a codimension-2 subvariety and if we assume for simplicity that the local defining equations f_1, f_2 for D_1, D_2 around V are regular, then $[V]$ occurs in both intersection products with the coefficient $l_A(A/(f_1, f_2))$, where $A = O_{X,V}$ is the local ring of X at V.

Case 2: X is a smooth scheme, so that Weil and Cartier divisors agree on X. Then it suffices to compare the intersection products $W \cdot V$ and $V \cdot W$ for any two $(n-1)$-dimensional subvarieties V, W of X. But the two products are obviously equal if $V = W$, and they are equal by case 1 if $V \neq W$. $\square$

Corollary 9.4.5. *The intersection product passes to rational equivalence, i. e. there are well-defined bilinear intersection maps $\operatorname{Pic} X \times A_k(X) \to A_{k-1}(X)$ determined by $D \cdot [V] = [D \cdot V]$ for all $D \in \operatorname{Pic} X$ and all k-dimensional subvarieties V of X.*

Proof. All that remains to be shown is that $D \cdot \alpha = 0$ for any Cartier divisor D if the cycle α is zero in the Chow group $A_k(X)$. But this follows from proposition 9.4.4, as for any rational function φ on a $(k+1)$-dimensional subvariety W of X we have

$$D \cdot [\operatorname{div}(\varphi)] = \operatorname{div}(\varphi) \cdot [D] = 0$$

(note that $\operatorname{div}(\varphi)$ is a Cartier divisor on W that is linearly equivalent to zero). $\square$

Remark 9.4.6. Obviously we can now iterate the process of taking intersection products with Cartier divisors: if X is a scheme and $D_1, \ldots, D_m$ are Cartier divisors (or divisor classes) on X then there are well-defined commutative intersection products

$$D_1 \cdot D_2 \cdots D_m \cdot \alpha \in A_{k-m}(X)$$

for any k-cycle $\alpha \in A_k(X)$. If X is an n-dimensional variety and $\alpha = [X]$ is the class of X we usually omit $[X]$ from the notation and write the intersection product simply as $D_1 \cdot D_2 \cdots D_m \in A_{n-m}(X)$. If $m = n$ and X is complete, the notation $D_1 \cdot D_2 \cdots D_m$ is moreover often used to denote the *degree* of the 0-cycle $D_1 \cdot D_2 \cdots D_m \in A_0(X)$ (see example 9.2.13) instead of the cycle itself. If a divisor D occurs m times in the intersection product we will also write this as D^m.

Example 9.4.7. Let $X = \mathbb{P}^2$. Then $\operatorname{Pic} X = A_1(X) = \mathbb{Z} \cdot [H]$, and the intersection product is determined by $H^2 = 1$ ("two lines intersect in one point"). In the same way, $H^n = 1$ on $\mathbb{P}^n$.

Example 9.4.8. Let $X = \tilde{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ in a point P. By example 9.2.14 we have $\operatorname{Pic} X = \mathbb{Z}[H] \oplus \mathbb{Z}[E]$, where E is the exceptional divisor, and H is a line in $\mathbb{P}^2$ not intersecting E. The strict transform L of a line in $\mathbb{P}^2$ through P has class $[L] = [H] - [E] \in \operatorname{Pic} X$.

The intersection products on X are therefore determined by computing the three products H^2, $H \cdot E$, and E^2. Of course, $H^2 = 1$ and $H \cdot E = 0$ (as $H \cap E = \emptyset$). To compute E^2 we use the relation $[E] = [H] - [L]$ and the fact that E and L meet in one point (with multiplicity 1):

$$E^2 = E \cdot (H - L) = E \cdot H - E \cdot L = 0 - 1 = -1.$$

By our interpretation of example 9.4.3 (ii) this means that the normal bundle of $E \cong \mathbb{P}^1$ in X is $\mathcal{O}_{\mathbb{P}^1}(-1)$. In particular, this normal bundle has no global sections. This means that E *cannot* be deformed in X as in the picture of example 9.4.3 (ii): one says that the curve E is *rigid* in X.

We can consider the formulas $H^2 = 1$, $H \cdot E = 0$, $E^2 = -1$, together with the existence of the intersection product $\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$ as a Bézout style theorem for the blow-up $X = \tilde{\mathbb{P}}^2$. In the same way, we get Bézout style theorems for other (smooth) surfaces and even higher-dimensional varieties.

Example 9.4.9. As a more complicated example, let us reconsider the question of exercise 4.6.6: how many lines are there in $\mathbb{P}^3$ that intersect four general given lines $L_1, \ldots, L_4 \subset \mathbb{P}^3$? Recall from exercise 3.5.4 that the space of lines in $\mathbb{P}^3$ is the smooth four-dimensional Grassmannian variety $X = G(1,3)$ that can be described as the set of all rank-2 matrices

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

modulo row transformations. By the Gaussian algorithm it follows that $G(1,3)$ has a stratification by affine spaces $X_4, X_3, X_2, X_2', X_1, X_0$ (where the subscript denotes the dimension and the stars denote arbitrary complex numbers)

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix} \qquad \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \qquad \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\quad X_4 \qquad\qquad\qquad X_3 \qquad\qquad\qquad X_2$$

$$\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\quad X_2' \qquad\qquad\qquad X_1 \qquad\qquad\qquad X_0$$

If we denote by $\sigma_4, \ldots, \sigma_0$ the classes in $A_*(X)$ of the closures of $X_4, \ldots, X_0$, we have seen in remark 9.1.18 that $A_*(X)$ is generated by the classes $\sigma_4, \ldots, \sigma_0$. These classes actually all have a geometric interpretation:

  (i) $\sigma_4 = [X]$.
  (ii) σ_3 is the class of all lines that intersect the line $\{x_0 = x_1 = 0\} \subset \mathbb{P}^3$. Note that this is precisely the zero locus of $a_0 b_1 - a_1 b_0$. In particular, if $L \subset \mathbb{P}^3$ is any other line then the class σ_3^L of all lines in $\mathbb{P}^3$ meeting L is also a quadratic function q in the entries of the matrix that is invariant under row transformations (in fact a 2×2 minor in a suitable choice of coordinates of $\mathbb{P}^3$). The quotient $\frac{a_0 b_1 - a_1 b_0}{q}$ is then a rational function on X whose divisor is $\sigma_3 - \sigma_3^L$. It follows that the class σ_3^L does not depend on L. So we can view σ_3 as the class that describes all lines intersecting *any* given line in $\mathbb{P}^3$.
  (iii) σ_2 is the class of all lines passing through the point $(0:0:0:1)$. By an argument similar to that in (ii) above, we can view σ_2 as the class of all lines passing through *any* given point in $\mathbb{P}^3$.
  (iv) σ_2' is the class of all lines that are contained in a plane (namely in the plane $x_0 = 0$ for the cycle X_2' given above).
  (v) σ_1 is the class of all lines that are contained in a plane and pass through a given point in this plane.
  (vi) σ_0 is the class of all lines passing through two given points in $\mathbb{P}^3$.

Hence we see that the intersection number we are looking for is just $\sigma_3^4 \in A_0(X) \cong \mathbb{Z}$ — the number of lines intersecting any four given lines in $\mathbb{P}^3$. So let us compute this number.

Step 1. Let us compute $\sigma_3^2 \in A_2(X)$, i.e. class of all lines intersecting two given lines L_1, L_2 in $\mathbb{P}^3$. We have seen above that it does not matter which lines we take, so let us choose L_1 and L_2 such that they intersect in a point $P \in \mathbb{P}^3$. A line that intersects both L_1 and L_2 has then two possibilities:

  (i) it is any line in the plane spanned by L_1 and L_2,
  (ii) it is any line in $\mathbb{P}^3$ passing through P.

As (i) corresponds to σ_2' and (ii) to σ_2 we see that $\sigma_3^2 = \sigma_2 + \sigma_2'$. To be more precise, we still have to show that σ_3^2 contains both X_2 and X_2' with multiplicity 1 (and not with a higher multiplicity). As an example, we will show that σ_3^2 contains σ_2 with multiplicity 1; the proof for σ_2' is similar. Consider the open subset $X_4 \subset G(1,3)$; it is isomorphic to an affine space $\mathbb{A}^4$ with coordinates a_2, a_3, b_2, b_3. On this open subset, the space of lines intersecting the line $\{x_0 = x_2 = 0\}$ is given scheme-theoretically by the equation $b_2 = 0$, whereas the space of lines intersecting the line $\{x_0 = x_3 = 0\}$ is given scheme-theoretically by the equation $b_3 = 0$. The scheme-theoretic intersection of these two spaces (i.e. the product σ_3^2) is then given by $b_2 = b_3 = 0$, which is precisely the locus of lines through the point $(0:1:0:0)$ (with multiplicity 1), i.e. the cycle σ_2.

Step 2. In the same way we compute that

  (i) $\sigma_3 \cdot \sigma_2 = \sigma_1$ (lines meeting a line L and a point P are precisely lines in the plane spanned by L and P passing through P),
  (ii) $\sigma_3 \cdot \sigma_2' = \sigma_1$ (lines meeting a line L and contained in a plane H are precisely lines in the plane H passing through the point $H \cap L$),
  (iii) $\sigma_3 \cdot \sigma_1 = \sigma_0$.

So we conclude that

$$\sigma_3^4 = \sigma_3^2(\sigma_2 + \sigma_2') = 2\sigma_3 \sigma_1 = 2,$$

i.e. there are exactly two lines in $\mathbb{P}^3$ meeting four other general given lines.

We should note that similar decompositions into affine spaces exist for all Grassmannian varieties, as well as rules how to intersect the corresponding Chow cycles. These rules are usually called *Schubert calculus*. They can be used to answer almost any question of the form: how many lines in $\mathbb{P}^n$ satisfy some given conditions?

Finally, let us prove a statement about intersection products that we will need in the next section. It is based on the following set-theoretic idea: let $f : X \to Y$ be any map of sets, and let $V \subset X$ and $W \subset Y$ be arbitrary subsets. Then it is checked immediately that

$$f(f^{-1}(W) \cap V) = W \cap f(V).$$

This relation is called a **projection formula**. There are projection formulas for many other morphisms and objects that can be pushed forward and pulled back along a morphism. We will prove an intersection-theoretic version here.

Lemma 9.4.10. *Let $f : X \to Y$ be a proper surjective morphism of schemes. Let $\alpha \in A_k(X)$ be a k-cycle on X, and let $D \in \operatorname{Pic} Y$ be a Cartier divisor (class) on Y. Then*

$$f_*(f^*D \cdot \alpha) = D \cdot f_*\alpha \in A_{k-1}(Y).$$

Proof. (Note that this is precisely the set-theoretic intersection formula from above, together with the statement that the scheme-theoretic multiplicities match up in the right way.)

By linearity we may assume that $\alpha = [V]$ for a k-dimensional subvariety $V \subset X$. Let $W = f(V)$, and denote by $g : V \to W$ the restriction of f to V. Then the left hand side of the equation of the lemma is $g_*[g^*D']$, where D' is the Cartier divisor on W associated to the line bundle $O_Y(D)|_W$. The right hand side is $[K(V) : K(W)] \cdot [D']$ by construction 9.2.9, with the convention that $[K(V) : K(W)] = 0$ if $\dim W < \dim V$. We will prove that these expressions actually agree in $Z_{k-1}(W)$ for any given Cartier divisor D'. This is a local statement (as we just have to check that every codimension-1 subvariety of W occurs on both sides with the same coefficient), so passing to an open subset we can assume that D' is the divisor of a rational function φ on W. But then by theorem 9.2.11 the left hand side is equal to

$$g_* \operatorname{div}(g^*\varphi) = \operatorname{div} N(g^*\varphi) = \operatorname{div}(\varphi^{[K(V):K(W)]}) = [K(V) : K(W)] \cdot \operatorname{div}(\varphi),$$

which equals the right hand side.   □

9.5. **Exercises.**

Exercise 9.5.1. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Compute the Chow group $A_{n-1}(\mathbb{P}^n \backslash X)$.

Exercise 9.5.2. Compute the Chow groups of $X = \mathbb{P}^n \times \mathbb{P}^m$ for all $n, m \geq 1$. Assuming that there are "intersection pairing homomorphisms"

$$A_{n+m-k}(X) \times A_{n+m-l}(X) \to A_{n+m-k-l}(X), \quad (\alpha, \alpha') \mapsto \alpha \cdot \alpha'$$

such that $[V \cap W] = [V] \cdot [W]$ for all subvarieties $V, W \subset X$ that intersect in the expected dimension, compute these homomorphisms explicitly. Use this to state a version of Bézout's theorem for products of projective spaces.

Exercise 9.5.3. (This is a generalization of example 9.1.7.) If X_1 and X_2 are closed subschemes of a scheme X show that there are exact sequences

$$A_k(X_1 \cap X_2) \to A_k(X_1) \oplus A_k(X_2) \to A_k(X_1 \cup X_2) \to 0$$

for all $k \geq 0$.

Exercise 9.5.4. Show that for any schemes X and Y there are well-defined product homo-morphisms
$$A_k(X) \times A_l(Y) \to A_{k+l}(X \times Y), \quad [V] \times [W] \mapsto [V \times W].$$
If X has a stratification by affine spaces as in remark 9.1.18 show that the induced homo-morphisms
$$\bigoplus_{k+l=m} A_k(X) \times A_l(Y) \to A_m(X \times Y)$$
are surjective. (In general, they are neither injective nor surjective).

Exercise 9.5.5. Prove the following criteria to determine whether a morphism $f : X \to Y$ is proper:

  (i) The composition of two proper morphisms is proper.
 (ii) Properness is "stable under base change": if $f : X \to Y$ is proper and $g : Z \to Y$ is any morphism, then the induced morphism $f' : X \times_Y Z \to Z$ is proper as well.
(iii) Properness is "local on the base": if $\{U_i\}$ is any open cover of Y and the restric-tions $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ are proper for all i then f is proper.
 (iv) Closed immersions (see 7.2.10) are proper.

Exercise 9.5.6. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the morphism given in homogeneous coordinates by $(x_0 : x_1) \mapsto (x_0^2 : x_1^2)$. Let $P \in \mathbb{P}^1$ be the point $(1 : 1)$, and consider the restriction $\tilde{f} : \mathbb{P}^1 \backslash \{P\} \to \mathbb{P}^1$. Show that $\tilde{f}$ is not proper, both with the topological and the algebraic definition of properness.

Exercise 9.5.7. For any $n > 0$ compute the Chow groups of $\mathbb{P}^2$ blown up in n points.

Exercise 9.5.8. Let k be an algebraically closed field. In this exercise we will construct an example of a variety that is complete (i. e. compact if $k = \mathbb{C}$) but not projective.

Consider $X = \mathbb{P}^3$ and the curves $C_1 = \{x_3 = x_2 - x_1 = 0\}$ and $C_2 = \{x_3 = x_0 x_2 - x_1^2 = 0\}$ in X. Denote by $P_1 = (1 : 0 : 0 : 0)$ and $P_2 = (1 : 1 : 1 : 0)$ their two intersection points.

Let $\tilde{X}_1' \to X$ be the blow-up at C_1, and let $\tilde{X}_1 \to \tilde{X}_1'$ be the blow-up at the strict trans-form of C_2. Denote by $\pi_1 : \tilde{X}_1 \to X$ the projection map. Similarly, let $\pi_2 : \tilde{X}_2 \to X$ be the composition of the two blow-ups in the opposite order; first blow up C_2 and then the strict transform of C_1. Obviously, $\tilde{X}_1$ and $\tilde{X}_2$ are isomorphic away from the inverse im-age of $\{P_1, P_2\}$, so we can glue $\pi_1^{-1}(X \backslash \{P_1\})$ and $\pi_2^{-1}(X \backslash \{P_2\})$ along the isomorphism $\pi_1^{-1}(X \backslash \{P_1, P_2\}) \cong \pi_2^{-1}(X \backslash \{P_1, P_2\})$ to get a variety Y. This variety will be our example. From the construction there is an obvious projection map $\pi : Y \to X$.

  (i) Show that Y is proper over k.
 (ii) For $i = 1, 2$ we know that C_i is isomorphic to $\mathbb{P}^1$. Hence we can choose a rational function φ_i on C_i with divisor $P_1 - P_2$. Compute the divisor of the rational function $\varphi_i \circ \pi$ on the variety $\pi^{-1}(C_i)$, as an element in $Z_1(Y)$.
(iii) From (ii) you should have found two irreducible curves $D_1, D_2 \subset Y$ such that $[D_1] + [D_2] = 0 \in A_1(Y)$. Deduce that Y is not a projective variety.

Exercise 9.5.9. Let X be a smooth projective surface, and let $C, D \subset X$ be two curves in X that intersect in finitely many points.

  (i) Prove that there is an exact sequence of sheaves on X
$$0 \to O_X(-C-D) \to O_X(-C) \oplus O_X(-D) \to O_X \to O_{C \cap D} \to 0.$$
 (ii) Conclude that the intersection product $C \cdot D \in \mathbb{Z}$ is given by the formula
$$C \cdot D = \chi(X, O_X) + \chi(X, O_X(-C-D)) - \chi(X, O_X(-C)) - \chi(X, O_X(D))$$
where $\chi(X, \mathcal{F}) = \sum_i (-1)^i h^i(X, \mathcal{F})$ denotes the Euler characteristic of the sheaf $\mathcal{F}$.

(iii) Show how the idea of (ii) can be used to define an intersection product of divisors on a smooth complete surface (even if the divisors do not intersect in dimension zero).