TOPOLOGICAL RECURSION RELATIONS AND GROMOV-WITTEN INVARIANTS IN HIGHER GENUS

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ABSTRACT. We state and prove a topological recursion relation that expresses any genus-g Gromov-Witten invariant of a projective manifold with at least a $(3g-1)$-st power of a cotangent line class in terms of invariants with fewer cotangent line classes. For projective spaces, we prove that these relations together with the Virasoro conditions are sufficient to calculate the full Gromov-Witten potential. This gives the first computationally feasible way to determine the higher genus Gromov-Witten invariants of projective spaces.

Consider a moduli space $\bar{M}_{g,n}(X,\beta)$ of n-pointed genus-g stable maps of class β to a complex projective manifold X. For any $1 \leq i \leq n$ let $\psi_i \in A^1(\bar{M}_{g,n}(X,\beta))$ be the i-th cotangent line class, i.e. the first Chern class of the line bundle on $\bar{M}_{g,n}(X,\beta)$ whose fibers are the cotangent spaces of the underlying curves at the i-th marked point. Equations in the Chow group of $\bar{M}_{g,n}(X,\beta)$ that express products of cotangent line classes in terms of boundary classes (i.e. classes on moduli spaces of *reducible* stable maps) are called topological recursion relations.

It has been proven recently by E. Ionel that any product of at least g cotangent line classes on $\bar{M}_{g,n}(X,\beta)$ is a sum of boundary cycles [I]. Unfortunately, the corresponding topological recursion relations are not yet known explicitly for general g. The $g \leq 2$ cases can be found in [Ge]. In theory, it should be possible to derive the equations for other (at least low) values of g from Ionel's work. As g grows however, the terms in the topological recursion relations become very complicated, and their number seems to grow exponentially. Consequently, Ionel's result is barely useful for actual computations, although it is of course very interesting from a theoretical point of view.

In this paper, we will prove a seemingly much weaker topological recursion relation that expresses only a product of at least $3g-1$ cotangent line classes at the same point in terms of boundary cycles. The idea to obtain this relation is simple: we just pull back the obvious relation $\psi_1^{3g-1} = 0$ on $\bar{M}_{g,1}$ to $\bar{M}_{g,n}(X,\beta)$ along the forgetful map, and keep track of the various pull-back correction terms in a clever way.

The result is a topological recursion relation that is extremely easy to state and apply. To be precise, denote by $\langle\langle \tau_{m_1}(\gamma_1) \cdots \tau_{m_n}(\gamma_n) \rangle\rangle_g$ the genus-g Gromov-Witten correlation function (i.e. the generating function for all invariants containing $ev_1^*\gamma_1 \cdot \psi_1^{m_1} \cdots ev_n^*\gamma_n \cdot \psi_n^{m_n}$; see section 1 for details). Let $\{T_a\}$ be a basis of the

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cohomology of X, and denote by $\{T^a\}$ the Poincaré-dual basis. Then for any $m \geq 0$

$$\langle\!\langle \tau_{3g-1+m}(\gamma) \rangle\!\rangle_g = \sum_{i+j=3g-2} \langle\!\langle \tau_m(\gamma) T_a \rangle\!\rangle^i \langle\!\langle \tau_j(T^a) \rangle\!\rangle_g,$$

where we use the Einstein summation convention over a, and where the auxiliary correlation functions $\langle\!\langle \cdots \rangle\!\rangle^i$ are defined recursively by

$$\langle\!\langle \tau_m(\gamma_1)\gamma_2 \rangle\!\rangle^i = \langle\!\langle \tau_{m+1}(\gamma_1)\gamma_2 \rangle\!\rangle^{i-1} - \langle\!\langle \tau_m(\gamma_1) T_a \rangle\!\rangle_0 \langle\!\langle T^a \gamma_2 \rangle\!\rangle^{i-1}$$

with the initial condition

$$\langle\!\langle \cdots \rangle\!\rangle^0 = \langle\!\langle \cdots \rangle\!\rangle_0.$$

Unlike other topological recursion relations, our relations involve neither sums over graphs nor invariants of genus other than g and 0. Moreover, the auxiliary functions $\langle\!\langle \cdots \rangle\!\rangle^i$ are "universal" in the sense that they do not depend on g. All this makes our relations very easy and fast to apply.

The application that we have in mind in this paper is the Virasoro conditions for the Gromov-Witten invariants of projective spaces. It has been proven recently by Givental that the Gromov-Witten potential of a projective space $\mathbb{P}^r$ satisfies an infinite series of differential equations, called the Virasoro conditions [Gi]. It is easily checked that these equations allow for recursion over the genus and the number of marked points in the following sense: given $g > 0$, $n \geq 1$, cohomology classes $\gamma_2, \ldots, \gamma_n \in A^*(X)$, and non-negative integers $m_2, \ldots, m_n$, the Virasoro conditions can express linear combinations of invariants

$$\langle \tau_m(\gamma) \tau_{m_2}(\gamma_2) \cdots \tau_{m_n}(\gamma_n) \rangle_{g,d}$$

(where $m \geq 0$, $\gamma \in A^*(X)$, and the degree $d \geq 0$ vary) in terms of other invariants with either smaller genus, or the same genus and smaller number of marked points. There is one such invariant for every choice of m, i.e. $r+1$ invariants for every choice of d. There is however only one non-trivial Virasoro condition for every d. Consequently, the Virasoro conditions alone are not sufficient to compute the Gromov-Witten invariants.

This is where the topological recursion relations come to our rescue. By inserting them into the Virasoro conditions, we can effectively bound the value of m in the set of unknown invariants above, leaving only the invariants with $0 \leq m < 3g-1$. This way we arrive at infinitely many linear Virasoro conditions (one for every choice of d) for only $3g-1$ invariants. It is now of course strongly expected that this system should be solvable, i.e. that the coefficient matrix of this system of linear equations has maximal rank $3g-1$. We will show that this is indeed always the case. In fact, we will show that *any* choice of $3g-1$ distinct non-trivial Virasoro conditions leads to a system of linear equations that determines the invariants uniquely. We do this by computing the determinant of the corresponding coefficient matrix: if we pick the Virasoro conditions associated to the degrees $d_0, \ldots, d_{3g-2}$ and reduce the cotangent line powers by our topological recursion relations, we arrive at a system of $3g-1$ linear equations for $3g-1$ invariants whose determinant is simply

$$\frac{\prod_{i>j}(d_i-d_j)}{\prod_{i=1}^{3g-2} i!} \cdot \prod_{i=1}^{3g-2} \left(i+\frac{1}{2}\right)^{3g-1-i},$$

which is obviously always non-zero. Therefore the Virasoro recursion works, i.e. we have found a constructive (and not too complicated) way to compute the Gromov-Witten invariants of $\mathbb{P}^r$ in any genus.

One should note that the result for the determinant above is remarkably simple, given the complicated structure of the Virasoro equations and our somewhat arbitrary choice of topological recursion relations. It would be interesting to see whether there is some deeper connection between the Virasoro conditions and the topological recursion relations that explains this easy result. It would also be interesting to extend our result to other Fano varieties.

We should also mention that our results have already been conjectured some time ago by Eguchi and Xiong [EX]. In their paper, they motivate our topological recursion relations by arguments from string theory. Assuming that these relations hold, Eguchi and Xiong use them together with the Virasoro conditions to compute a few examples of higher genus Gromov-Witten invariants of projective spaces. From this point of view one can regard our paper as providing a solid mathematical footing for [EX].

The paper is organized as follows. In section 1 we will establish the topological recursion relations mentioned above. We will then describe in section 2 how to apply these results to the Virasoro conditions to get systems of linear equations for the Gromov-Witten invariants of projective spaces. The proof that these systems of equations are always solvable (i.e. the computation of the determinant mentioned above) is given in section 3. Finally, we will list some numbers obtained with our method in section 4.

A C++ program that implements the algorithm of our paper and computes the Gromov-Witten invariants of projective spaces can be obtained from the author on request.

1. Topological recursion relations

The goal of this section is to prove the topological recursion relation stated in the introduction. To do so, we will first compare certain cycles in the moduli spaces of stable and prestable curves.

For any $g,n \geq 0$ let $\bar{\mathfrak{M}}_{g,n}$ be the moduli space of complex n-pointed genus-g prestable curves, i.e. the moduli space of tuples $(C,x_1,\dots,x_n)$ where C is a nodal curve of arithmetic genus g, and the x_i are distinct smooth points of C. This is a proper (but not separated) smooth Artin stack of dimension $3g-3+n$. The open substack corresponding to irreducible curves is denoted $\mathfrak{M}_{g,n}$.

Recall that a prestable curve $(C,x_1,\dots,x_n)$ is called stable if every rational component has at least 3 and every elliptic component at least 1 special point, where the special points are the nodes and the marked points. The open substack of $\bar{\mathfrak{M}}_{g,n}$ corresponding to stable curves is denoted $\bar{M}_{g,n}$. It is a proper, separated, smooth Deligne-Mumford stack. If $2g+n \geq 3$ there is a stabilization morphism $s : \bar{\mathfrak{M}}_{g,n} \to \bar{M}_{g,n}$ that contracts every unstable component.

On any of the above moduli spaces and for any of the marked points x_i we define the cotangent line class, denoted ψ_{x_i}, to be the first Chern class of the line bundle on the moduli space whose fiber at the point $(C, x_1, \dots, x_n)$ is the cotangent space $T^\vee_{C,x_i}$. Cotangent line classes do not remain unchanged under stabilization — they receive correction terms from the locus of reducible curves where the marked point is on an unstable component. Our first task is therefore to compare the cycles $s^*\psi_{x_i}$ and ψ_{x_i} on $\bar{\mathfrak{M}}_{g,n}$ for $g > 0$. For simplicity, we will do this here only in the case $n = 1$. As there is then only one cotangent line class, we will simply write it as ψ (with no index).

Let us define the "correction terms" that we will pick up when pulling back cotangent line classes. For fixed $g > 0$ and any $k \geq 0$ let M_k be the product

$$M_k = \bar{\mathfrak{M}}_{g,1} \times \underbrace{\bar{\mathfrak{M}}_{0,2} \times \cdots \times \bar{\mathfrak{M}}_{0,2}}_{k \text{ copies}}.$$

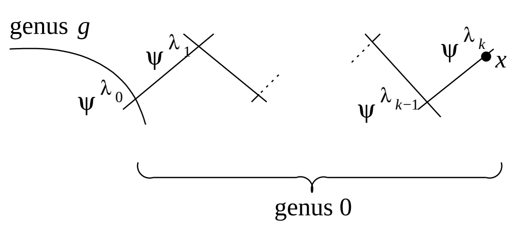
Obviously, points of M_k correspond to a collection $(C^{(0)}, \dots, C^{(k)})$ of prestable curves with $C^{(0)}$ of genus g and $C^{(i)}$ of genus 0 for $i > 0$, together with marked points x_0 on $C^{(0)}$ and x_i, y_i on $C^{(i)}$ for $i > 0$.

There is a natural proper gluing morphism $\pi : M_k \to \bar{\mathfrak{M}}_{g,1}$ that sends any point $(C^{(0)}, \dots, C^{(k)}, x_0, x_1, y_1, \dots, x_k, y_k)$ to the nodal 1-pointed curve $(C^{(0)} \cup \cdots \cup C^{(k)}, x_k)$, where $C^{(i-1)}$ is glued to $C^{(i)}$ for all $i = 1, \dots, k$ by identifying x_{i-1} with y_i. The morphism π is an isomorphism onto its image on the open subset $\mathfrak{M}_{g,1} \times \mathfrak{M}_{0,2} \times \cdots \times \mathfrak{M}_{0,2}$. Obviously, a generic point in the image of π is just a curve with $k+1$ components aligned in a chain, with the first component having genus g, and all others being rational. The marked point is always on the last component.

For any collection $\lambda = (\lambda_0, \dots, \lambda_k)$ of $k+1$ non-negative integers we denote by $Z(\lambda)$ the cycle

$$\pi_*(\psi_{x_0}^{\lambda_0} \cdots \psi_{x_k}^{\lambda_k} \cdot [M_k]) \in A_*(\bar{\mathfrak{M}}_{g,1})$$

in the Chow group of $\bar{\mathfrak{M}}_{g,1}$ (for intersection theory on Artin stacks we refer to [K]). The cycle $Z(\lambda)$ has pure codimension $k+\lambda_0+\cdots+\lambda_k$ in $\bar{\mathfrak{M}}_{g,1}$. It can be represented graphically as



where the cotangent line classes sit at the "left" points of the nodes (except for the last one that sits on the remaining marked point).

With these cycles $Z(\lambda)$ we can now formulate the pull-back transformation rule for cotangent line classes.

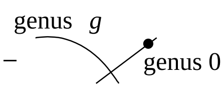
Lemma 1.1. *For any $k \geq 0$ and $\lambda_0, \dots, \lambda_k \geq 0$ we have*

$$s^*\psi \cdot Z(\lambda_0, \dots, \lambda_k) = Z(\lambda_0 + 1, \lambda_1, \dots, \lambda_k) - Z(0, \lambda_0, \lambda_1, \dots, \lambda_k)$$

in $A_*(\bar{\mathfrak{M}}_{g,1})$, where $s : \bar{\mathfrak{M}}_{g,1} \to \bar{M}_{g,1}$ is the stabilization map.

Proof. Let $p_1 : M_k = \bar{\mathfrak{M}}_{g,1} \times \bar{\mathfrak{M}}_{0,2} \times \cdots \times \bar{\mathfrak{M}}_{0,2} \to \bar{\mathfrak{M}}_{g,1}$ be the projection onto the first factor. Note that $s \circ p_1 = s \circ \pi$.

By [Ge] proposition 5 we have the transformation rule

$s^*\psi \;=\; \psi \;-$ 

on $\bar{\mathfrak{M}}_{g,1}$. Pulling this equation back by p_1, intersecting with $\psi_{x_0}^{\lambda_0} \cdots \psi_{x_k}^{\lambda_k}$, and pushing it forward again by π then gives the equation of the lemma. $\square$

It is easy to iterate this lemma to get an expression for $(s^*\psi)^N$:

Corollary 1.2. *For all $N \geq 0$ we have*

$$(s^*\psi)^N = \sum_{k \geq 0} (-1)^k \sum_{\substack{(\lambda_0, \dots, \lambda_k) \\ k + \lambda_0 + \cdots + \lambda_k = N}} Z(\lambda_0, \lambda_1, \dots, \lambda_k)$$

in $A_*(\bar{\mathfrak{M}}_{g,1})$. In particular, the right hand side is zero if $N \geq 3g - 1$.

Proof. The statement is obvious for $N = 0$ as $\bar{\mathfrak{M}}_{g,1} = Z(0)$. The equation now follows immediately by induction from lemma 1.1. Moreover, note that $\bar{M}_{g,1}$ is a Deligne-Mumford stack of dimension $3g - 2$, so its Chow groups in codimension at least $3g - 1$ vanish. Therefore $(s^*\psi)^N = s^*(\psi^N) = 0$ for $N \geq 3g - 1$. $\square$

Remark 1.3. Note that in the spaces M_k the marked point is always on the last component. So by intersecting the equation of corollary 1.2 with the m-th power of the cotangent line class on $\bar{\mathfrak{M}}_{g,1}$ we get

$$\psi^m \cdot (s^*\psi)^N = \sum_{k \geq 0} (-1)^k \sum_{\substack{(\lambda_0, \dots, \lambda_k) \\ k + \lambda_0 + \cdots + \lambda_k = N}} Z(\lambda_0, \lambda_1, \dots, \lambda_{k-1}, \lambda_k + m)$$

in $A_*(\bar{\mathfrak{M}}_{g,1})$ for all $N, m \geq 0$. As in the corollary, the right hand side will be zero if $N \geq 3g - 1$.

We will now apply this result to moduli spaces of stable maps. So let X be a complex projective manifold, and let β be the homology class of an algebraic curve in X. As usual we denote by $\bar{M}_{g,n}(X, \beta)$ the moduli space of n-pointed genus-g stable maps of class β to X (see e.g. [FP]). It is a proper Deligne-Mumford stack of virtual dimension

$$\operatorname{vdim} \bar{M}_{g,n}(X, \beta) = -K_X \cdot \beta + (\dim X - 3)(1 - g) + n.$$

The actual dimension of $\bar{M}_{g,n}(X, \beta)$ might (and for $g > 0$ usually will) be bigger than this virtual dimension. There is however always a canonically defined virtual fundamental class

$$[\bar{M}_{g,n}(X, \beta)]^{\mathrm{virt}} \in A_{\operatorname{vdim} \bar{M}_{g,n}(X,\beta)}(\bar{M}_{g,n}(X, \beta))$$

that is used instead of the true fundamental class in intersection theory, and that therefore makes the moduli space appear to have the "correct" dimension for intersection-theoretic purposes (see e.g. [BF], [B]).

The points in this moduli space can be written as $(C, x_1, \dots, x_n, f)$, where $(C, x_1, \dots, x_n) \in \bar{\mathfrak{M}}_{g,n}$, $f : C \to X$ is a morphism of degree β, and every rational (resp. elliptic) component *on which f is constant* has at least 3 (resp. 1) special points.

The moduli spaces $\bar{M}_{g,n}(X,\beta)$ come equipped with cotangent line classes ψ_{x_i} in the same way as above. In addition, for every marked point x_i we have an evaluation morphism $ev_{x_i} : \bar{M}_{g,n}(X,\beta) \to X$ given by $ev_{x_i}(C, x_1, \dots, x_n, f) = f(x_i)$. For any cohomology classes $\gamma_1, \dots, \gamma_n \in A^*(X)$ and non-negative integers $m_1, \dots, m_n$ we define the Gromov-Witten invariant

$$\langle \tau_{m_1}(\gamma_1) \cdots \tau_{m_n}(\gamma_n) \rangle_{g,\beta} := \int_{[\bar{M}_{g,n}(X,\beta)]^{\text{virt}}} \psi_{x_1}^{m_1} \cdot ev_{x_1}^* \gamma_1 \cdots \psi_{x_n}^{m_n} \cdot ev_{x_n}^* \gamma_n \in \mathbb{Q},$$

where the integral is understood to be zero if the integrand is not of dimension $\operatorname{vdim} \bar{M}_{g,n}(X,\beta)$. If $m_i = 0$ for some i we abbreviate $\tau_{m_i}(\gamma_i)$ as γ_i within the brackets on the left hand side. As the Gromov-Witten invariants are multilinear in the γ_i, it suffices to pick the γ_i from among a fixed basis. So let us choose a basis $\{T_a\}$ of the cohomology (modulo numerical equivalence) of X and let $\{T^a\}$ be the Poincaré-dual basis.

Remark 1.4. It is often convenient to encode the Gromov-Witten invariants as the coefficients of a generating function. So we introduce the so-called correlation functions

$$\langle\langle \tau_{m_1}(\gamma_1) \cdots \tau_{m_n}(\gamma_n) \rangle\rangle_g := \sum_\beta \left\langle \tau_{m_1}(\gamma_1) \cdots \tau_{m_n}(\gamma_n) \exp\left(\sum_m t_m^a \tau_m(T_a) \right) \right\rangle_{g,\beta} q^\beta$$

where the t_m^a and q^β are formal variables satisfying $q^{\beta_1} q^{\beta_2} = q^{\beta_1+\beta_2}$. Here and in the following we use the summation convention for the "cohomology index" a, i.e. an index occurring both as a lower and upper index is summed over. The correlation functions are formal power series in the variables t_m^a and q^β whose coefficients describe all genus-g Gromov-Witten invariants containing at least the classes $\tau_{m_1}(\gamma_1) \cdots \tau_{m_n}(\gamma_n)$.

With this notation we can now rephrase corollary 1.2 in terms of correlation functions for Gromov-Witten invariants:

Proposition 1.5. *For all $g > 0$, $N \geq 3g-1$, $m \geq 0$, and $\gamma \in A^*(X)$ we have*

$$0 = \sum_{k \geq 0} (-1)^k \sum_{\substack{(\lambda_0, \dots, \lambda_k) \\ k + \lambda_0 + \cdots + \lambda_k = N}} \langle\langle \tau_{\lambda_0}(T^{a_1}) \rangle\rangle_g \langle\langle T_{a_1} \tau_{\lambda_1}(T^{a_2}) \rangle\rangle_0 \cdots \langle\langle T_{a_{k-1}} \tau_{\lambda_{k-1}}(T^{a_k}) \rangle\rangle_0 \langle\langle T_{a_k} \tau_{\lambda_k + m}(\gamma) \rangle\rangle_0$$

as power series in t_m^a and q^β.

Proof. For every $n \geq 1$ and any homology class β there is a forgetful morphism

$$q : \bar{M}_{g,n}(X,\beta) \to \bar{\mathfrak{M}}_{g,1}, \quad (C, z_1, \dots, z_n, f) \mapsto (C, z_1).$$

(We denote the marked points by z_i in order not to confuse them with the x_i and y_i above that are used to glue the components of the reducible curves.) We claim that the statement of the proposition is obtained by pulling back the equation of remark 1.3 by q in the case $N \geq 3g-1$ and evaluating the result on the virtual fundamental class of $\bar{M}_{g,n}(X,\beta)$.

To show this, we obviously have to compute $q^* Z(\lambda)$ for all $\lambda = (\lambda_0, \dots, \lambda_k)$. Let $B = (\beta_0, \dots, \beta_k)$ be a collection of homology classes with $\sum_i \beta_i = \beta$, and let $I = (I_0, \dots, I_k)$ be a collection of subsets of $\{2, \dots, n\}$ whose union is $\{2, \dots, n\}$. Set

$$M_{B,I} = \bar{M}_{g,1+\#I_0}(X,\beta_0) \times_X \bar{M}_{0,2+\#I_1}(X,\beta_1) \times_X \cdots \times_X \bar{M}_{0,2+\#I_k}(X,\beta_k),$$

where the fiber products are taken over the evaluation maps at the first marked point of the $(i-1)$-st factor and the second marked point of the i-th factor for $i = 1, \dots, k$. In other words, the moduli space $M_{B,I}$ describes stable maps with the same configuration of components as in M_k, and with the homology class β and the marked points $z_2, \dots, z_n$ split up onto the components in a prescribed way (the point z_1 is always the first marked point of the last factor). Note that the space $M_{B,I}$ carries a natural virtual fundamental class induced from their factors. By [B] axiom III it is equal to the product cycle of the virtual fundamental classes of the factors, intersected with the pull-backs of the diagonal classes $\Delta_X \subset X \times X$ along the evaluation maps at every pair of marked points where two components are glued together.

Now by [B] axiom V we have a Cartesian diagram

$$\begin{tikzcd} \coprod_{B,I} M_{B,I} \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{q}"'] & \bar{M}_{g,n}(X,\beta) \arrow[d, "q"] \\ M_k \arrow[r, "\pi"] & \bar{\mathfrak{M}}_{g,1} \end{tikzcd}$$

with $\pi^![\bar{M}_{g,n}(X,\beta)]^{\mathrm{virt}} = \sum_{B,I} [M_{B,I}]^{\mathrm{virt}}$. As $\tilde{q}$ does not change the cotangent line classes we conclude that

$$\begin{aligned} q^* Z(\lambda) \cdot [\bar{M}_{g,n}(X,\beta)]^{\mathrm{virt}} &= q^* \pi_* (\psi_{x_0}^{\lambda_0} \cdots \psi_{x_k}^{\lambda_k}) \cdot [\bar{M}_{g,n}(X,\beta)]^{\mathrm{virt}} \\ &= \tilde{\pi}_* \tilde{q}^* (\psi_{x_0}^{\lambda_0} \cdots \psi_{x_k}^{\lambda_k}) \cdot [\bar{M}_{g,n}(X,\beta)]^{\mathrm{virt}} \\ &= \sum_{B,I} \tilde{\pi}_* \left(\psi_{x_0}^{\lambda_0} \cdots \psi_{x_k}^{\lambda_k} \cdot [M_{B,I}]^{\mathrm{virt}} \right), \end{aligned} \qquad (*)$$

where x_i denotes the first marked point of the i-th factor. Now choose cohomology classes $\gamma_1, \dots, \gamma_n$ and non-negative integers $m_1, \dots, m_n$. Intersecting expression $(*)$ with $ev_{z_1}^* \gamma_1 \cdot \psi_{z_1}^{m_1} \cdots ev_{z_n}^* \gamma_n \cdot \psi_{z_n}^{m_n}$ and taking the degree of the resulting homology class (if it is zero-dimensional), we get exactly the Gromov-Witten invariants

$$\sum_{B,I} \langle \tau_{\lambda_0}(T^{a_1}) \mathcal{T}_0 \rangle_{g,\beta_0} \langle T_{a_1} \tau_{\lambda_1}(T^{a_2}) \mathcal{T}_1 \rangle_{0,\beta_1} \cdots \langle T_{a_k} \tau_{\lambda_k + l_1}(\gamma_1) \mathcal{T}_k \rangle_{0,\beta_k},$$

where $\mathcal{T}_i$ is a short-hand notation for $\prod_{j \in I_i} \tau_{m_j}(\gamma_j)$. If we choose $\gamma_i = T^{a_i}$ for some a_i, this can obviously be rewritten as the $\big(q^\beta \cdot \prod_{i=1}^n t_{m_i}^{a_i}\big)$-coefficient of the function

$$\langle\langle \tau_{\lambda_0}(T^{a_1}) \rangle\rangle_g \langle\langle T_{a_1} \tau_{\lambda_1}(T^{a_2}) \rangle\rangle_0 \cdots \langle\langle T_{a_{k-1}} \tau_{\lambda_{k-1}}(T^{a_k}) \rangle\rangle_0 \langle\langle T_{a_k} \tau_{\lambda_k+m}(\gamma) \rangle\rangle_0.$$

Inserting this into the formula of remark 1.3 gives the desired result. □

Corollary 1.6. ***(Topological recursion relation)*** *For all $g > 0$, $N \geq 3g-1$, $m \geq 0$, and $\gamma \in A^*(X)$ we have*

$$\langle\langle \tau_{N+m}(\gamma) \rangle\rangle_g = \sum_{i+j=N-1} \langle\langle \tau_m(\gamma) T_a \rangle\rangle^i \langle\langle \tau_j(T^a) \rangle\rangle_g,$$

where the auxiliary correlation functions $\langle\langle \cdots \rangle\rangle^i$ are defined recursively by

$$\langle\langle \tau_m(\gamma_1)\gamma_2 \rangle\rangle^i = \langle\langle \tau_{m+1}(\gamma_1)\gamma_2 \rangle\rangle^{i-1} - \langle\langle \tau_m(\gamma_1) T_a \rangle\rangle_0 \langle\langle T^a \gamma_2 \rangle\rangle^{i-1}$$

with the initial condition

$$\langle\langle \cdots \rangle\rangle^0 = \langle\langle \cdots \rangle\rangle_0.$$

Proof. Note that the $k=0$ term in lemma 1.5 is just $\langle\langle \tau_{N+m}(\gamma) \rangle\rangle_g$. So we find that the equation of the corollary is true if we set

$$\langle\langle \tau_m(\gamma_1)\gamma_2 \rangle\rangle^i = \sum_{k \geq 0} (-1)^k \sum_{\substack{(\lambda_1,\ldots,\lambda_k) \\ k+\lambda_1+\cdots+\lambda_k = N}} \langle\langle T_{a_1} \tau_{\lambda_1}(T^{a_2}) \rangle\rangle_0 \cdots \langle\langle T_{a_{k-1}} \tau_{\lambda_{k-1}}(T^{a_k}) \rangle\rangle_0 \langle\langle T_{a_k} \tau_{\lambda_k+m}(\gamma) \rangle\rangle_0.$$

It is checked immediately that these correlation functions satisfy the recursive relations stated in the corollary. □

Remark 1.7. As in the case of the Gromov-Witten invariants, we will expand the correlation functions $\langle\langle \cdots \rangle\rangle^i$ as a power series in q^β and t_m^a and call the resulting coefficients $\langle \cdots \rangle_\beta^i$ according to the formula

$$\langle\langle \tau_{m_1}(\gamma_1)\gamma_2 \rangle\rangle^i = \sum_\beta \left\langle \tau_{m_1}(\gamma_1)\gamma_2 \exp\left(\sum_m t_m^a \tau_m(T_a)\right) \right\rangle_\beta^i q^\beta.$$

Note however that, in contrast to the Gromov-Witten numbers, the invariants $\langle \cdots \rangle^i$ must have at least two entries, of which the second one contains no cotangent line class.

For future computations it is convenient to construct a minor generalization of corollary 1.6 that is mostly notational. Note that all genus-0 degree-0 invariants with fewer than 3 marked points are trivially zero, as the moduli spaces of stable maps are empty in this case. It is an important and interesting fact that many formulas concerning Gromov-Witten invariants get easier if we assign “virtual values” to these invariants in the unstable range:

Convention 1.8. *Unless stated otherwise, we will from now on allow formal negative powers of the cotangent line classes (i.e. the index m in the $\tau_m(\gamma)$ can be any integer). Invariants $\langle \cdots \rangle_{g,\beta}$ and $\langle \cdots \rangle_\beta^i$ are simply defined to be zero if they contain a negative power of a cotangent line class at any point, except for the following cases of genus-0 degree-0 invariants with fewer than 3 marked points:*

(i) $\langle \tau_{-2}(\mathrm{pt}) \rangle_{0,0} = 1$,
(ii) $\langle \tau_{m_1}(\gamma_1)\tau_{m_2}(\gamma_2) \rangle_{0,0} = (-1)^{\max(m_1,m_2)}(\gamma_1 \cdot \gamma_2)\delta_{m_1+m_2,-1}$,
(iii) $\langle \tau_{-i-1}(\gamma_1)\gamma_2 \rangle_0^i = (\gamma_1 \cdot \gamma_2)$ *for all* $i \geq 0$.

The correlation functions $\langle\!\langle \cdots \rangle\!\rangle$ *are changed accordingly so that the equations of remarks 1.4 and 1.7 remain true (in particular these functions will now depend additionally on the variables* t_m^a *for* $m < 0$*).*

Remark 1.9. Note that this convention is consistent with the general formula for genus-0 degree-0 invariants

$$\langle \tau_{m_1}(\gamma_1) \cdots \tau_{m_n}(\gamma_n) \rangle_{0,0} = \binom{n-3}{m_1, \dots, m_n} (\gamma_1 \cdots \gamma_n)\delta_{m_1+\cdots+m_n,n-3},$$

as well as with the recursion relations for the $\langle\!\langle \cdots \rangle\!\rangle^i$ of corollary 1.6.

Using this convention, we can now restate our topological recursion relations as follows:

Corollary 1.10. *For all* $g > 0$, $N \geq 3g-1$, $m \in \mathbb{Z}$, *and* $\gamma \in A^*(X)$ *we have*

$$\langle\!\langle \tau_{N+m}(\gamma) \rangle\!\rangle_g = \sum_{i+j=N-1} \langle\!\langle \tau_m(\gamma)T_a \rangle\!\rangle^i \langle\!\langle \tau_j(T^a) \rangle\!\rangle_g,$$

where the auxiliary correlation functions $\langle\!\langle \cdots \rangle\!\rangle^i$ *are defined recursively by the formulas given in corollary 1.6, together with convention 1.8.*

Proof. The equations in the corollary are the same as in corollary 1.6 if $m \geq 0$. For $m < 0$ they reduce to the trivial equations $\langle\!\langle \tau_{N+m}(\gamma) \rangle\!\rangle_g = \langle\!\langle \tau_{N+m}(\gamma) \rangle\!\rangle_g$ by convention 1.8. $\square$

2. The Virasoro conditions

We now want to apply our topological recursion relation in conjunction with the Virasoro conditions to compute Gromov-Witten invariants. The Virasoro conditions are certain relations among Gromov-Witten invariants conjectured in [EHX] that have recently been proven for projective spaces by Givental [Gi]. We will therefore from now on restrict to the case $X = \mathbb{P}^r$. It is expected that the same methods would work for other Fano varieties as well.

To state the Virasoro conditions we need some notation. We pick the obvious basis $\{T_a\}$ of $A^*(X)$ where T_a denotes the class of a linear subspace of codimension a for $a = 0, \dots, r$. Let $R : A^*(X) \to A^*(X)$ be the homomorphism of multiplication with the first Chern class $c_1(X)$. In our basis, the p-th power R^p of R is then given by $(R^p)_a{}^b = (r+1)^p\, \delta_{a+p,b}$.

For any $x \in \mathbb{Q}$, $k \in \mathbb{Z}_{\geq -1}$, and $0 \leq p \leq k+1$ denote by $[x]_p^k$ the z^p-coefficient of $\prod_{j=0}^k (z+x+j)$, or in other words the $(k+1-p)$-th elementary symmetric polynomial in $k+1$ variables evaluated at the numbers $x, \dots, x+k$.

Then the Virasoro conditions state that for any $k \geq 1$ and $g \geq 1$ we have an equation of power series in t_m^a and q^β (see e.g. [EHX])

$$\begin{aligned}
0 = &- \sum_{p=0}^{k+1} \left[\frac{3-r}{2}\right]_p^k (R^p)_0{}^b \langle\!\langle \tau_{k+1-p}(T_b) \rangle\!\rangle_g && \text{(A)}\\
&+ \sum_{p=0}^{k+1} \sum_{m=0}^{\infty} \left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b t_m^a \langle\!\langle \tau_{k+m-p}(T_b) \rangle\!\rangle_g && \text{(B)}\\
&+ \frac{1}{2} \sum_{p=0}^{k+1} \sum_{m=p-k}^{-1} (-1)^m \left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b \langle\!\langle \tau_{-m-1}(T^a) \tau_{k+m-p}(T_b) \rangle\!\rangle_{g-1} && \text{(C)}\\
&+ \frac{1}{2} \sum_{p=0}^{k+1} \sum_{m=p-k}^{-1} \sum_{h=0}^{g} (-1)^m \left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b \langle\!\langle \tau_{-m-1}(T^a) \rangle\!\rangle_h \langle\!\langle \tau_{k+m-p}(T_b) \rangle\!\rangle_{g-h}, && \text{(D)}
\end{aligned}$$

where convention 1.8 is not yet applied (i.e. the genus-0 degree-0 invariants in the unstable range are defined to be zero). We should also mention that there are versions of these relations also for $k \geq -1$ and all $g \geq 0$, but the equations will then get additional correction terms that we have dropped here for the sake of simplicity.

First of all let us apply convention 1.8 to these formulas. It is checked immediately that this realizes the (A) and (B) terms as part of the (D) terms via the conventions (i) and (ii), respectively. So by applying our convention we can drop the (A) and (B) terms above (if we allow arbitrary integers in the sum over m).

Let us now analyze how these equations can be used to compute Gromov-Witten invariants. First of all we will compute the invariants recursively over the genus of the curves. The genus-0 invariants of $\mathbb{P}^r$ are well-known and can be computed by the WDVV equations (see e.g. the "first reconstruction theorem" of Kontsevich and Manin [KM]). So let us assume that we want to compute the invariants of some genus $g > 0$, and that we already know all invariants of smaller genus. In the Virasoro equations above this means that we know all of (C), as well as the terms of (D) where $h \neq 0$ and $h \neq g$. Noting that the (D) terms are symmetric under $h \mapsto g-h$, we can therefore rewrite the Virasoro conditions as

$$\begin{aligned}
\sum_{p=0}^{k+1} \sum_{m} (-1)^m \left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b \langle\!\langle \tau_{-m-1}(T^a) \rangle\!\rangle_0 \langle\!\langle \tau_{k+m-p}(T_b) \rangle\!\rangle_g \\
= \text{(recursively known terms)}.
\end{aligned}$$

Next, we will compute the invariants of genus g recursively over the number of marked points. So let us assume that we want to compute the n-point genus-g invariants, and that we already know all invariants of genus g with fewer marked points. In the equations above this means that we fix a degree $d \geq 0$, integers $m_2, \dots, m_n$, and $n-1$ cohomology classes $T_{a_2}, \dots, T_{a_n}$, and compare the $\big(q^d \cdot \prod_{i=2}^n t_{m_i}^{a_i}\big)$-coefficients of the equations. By the recursion process we then know all the invariants in which at least one of the marked points $x_2, \dots, x_n$ is on the genus-0 invariant. So we can write

$$\sum_{p=0}^{k+1}\sum_{m}\sum_{d_1+d_2=d}(-1)^m\left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b\cdot$$
$$\cdot\langle\tau_{-m-1}(T^a)\rangle_{0,d_1}\langle\tau_{k+m-p}(T_b)\tau_{m_2}(T_{a_2})\cdots\tau_{m_n}(T_{a_n})\rangle_{g,d_2}$$
$$=(\text{recursively known terms}).$$

These are equations for the unknown invariants $\langle\tau_j(T_b)\tau_{m_2}(T_{a_2})\cdots\tau_{m_n}(T_{a_n})\rangle_{g,e}$, where j, b, and e vary. Note that for a given $j\geq 0$ there is exactly one such invariant $\langle\tau_j(T_{b_j})\tau_{m_2}(T_{a_2})\cdots\tau_{m_n}(T_{a_n})\rangle_{g,e_j}$: the values of b_j and e_j are determined uniquely by the dimension condition

$$(r+1)e_j+(r-3)(1-g)+n=j+b_j+\sum_{i=2}^{n}(m_i+a_i) \tag{1}$$

as we must have $0\leq b_j\leq r$. Let us denote this invariant by x_j. Of course it may happen that $e_j<0$, in which case we set $x_j=0$. Our equations now read

$$\sum_{p=0}^{k+1}\sum_{m}(-1)^{m+p-k}\left[a+m+p-k+\frac{1-r}{2}\right]_p^k (R^p)_a{}^{b_m}\langle\tau_{-m-p+k-1}(T^a)\rangle_{0,d-e_m}\cdot x_m$$
$$=(\text{recursively known terms}).$$

Let us now check how many non-trivial equations of this sort we get. Together with (1), the dimension conditions

$$(d-e_m)(r+1)+r-3+1=-m-p+k-1+r-a$$

(for the genus-0 invariant) and $a+p=b_m$ (from the R^p factor) give

$$k=d(r+1)+(r-3)(1-g)+n-1-\sum_{i=2}^{n}(m_i+a_i), \tag{2}$$

which means that the value of k is determined by d. To avoid overly complicated notation, in what follows we will denote the number k determined by (2) by $k(d)$. Moreover, let δ be the smallest value of d for which $k(d)$ is positive. We are then getting one equation for every degree $d\geq\delta$. As there are $r+1$ unknown invariants x_j in every degree however, it is clear that our equations alone are not sufficient to determine the x_j.

Let us now apply our topological recursion relations. In terms of the recursion at hand, these relations can express every invariant x_m as a linear combination of invariants of the same form with $m<3g-1$, plus some terms that are known recursively because they contain only invariants with fewer than n marked points. More precisely, we have

$$x_m=\sum_{i+j=N-1}\langle\tau_{m-N+2}(T_{b_m})T^{b_j}\rangle^i_{e_m-e_j}x_j+(\text{recursively known terms})$$

for all $N \geq 3g-1$ by corollary 1.10. Inserting this into the Virasoro conditions, we get

$$\sum_{p=0}^{k(d)+1} \sum_{m} \sum_{i+j=N-1} (-1)^{m+p-k(d)} \left[a+m+p-k(d)+\frac{1-r}{2}\right]_p^{k(d)} (R^p)_a{}^{b_m} \cdot \\ \cdot \langle \tau_{-m-p+k(d)-1}(T^a) \rangle_{0,d-e_m} \cdot \langle \tau_{m-N+2}(T_{b_m}) T^{b_j} \rangle^i_{e_m-e_j} \cdot x_j \\ = (\text{recursively known terms}).$$

Using the dimension conditions again, and noting that the sum over m is equivalent to independent sums over b_m and e_m, we can rewrite this as

$$\sum_{p=0}^{k(d)+1} \sum_{e} \sum_{i+j=N-1} (-1)^{1-a-(d-e)(r+1)} \left[\frac{3-r}{2}-(d-e)(r+1)\right]_p^{k(d)} (R^p)_a{}^{b} \cdot \\ \cdot \langle \tau_\bullet(T^a) \rangle_{0,d-e} \cdot \langle \tau_\bullet(T_b) T^{b_j} \rangle^i_{e-e_j} \cdot x_j \\ = (\text{recursively known terms}),$$

where the dots in the τ functions denote the uniquely determined numbers so that the invariants satisfy the dimension condition.

We are thus left with infinitely many equations (one for every $d \geq \delta$) for finitely many variables $x_0, \dots, x_{N-1}$. It is of course strongly expected that this system of equations should be solvable, i.e. that the matrix $V^{(N)} = (V^{(N)}_{d,j})_{d \geq \delta, 0 \leq j < N}$ with

$$V^{(N)}_{d,j} := \sum_{p=0}^{k(d)+1} \sum_{e} (-1)^{1-a-(d-e)(r+1)} \left[\frac{3-r}{2}-(d-e)(r+1)\right]_p^{k(d)} (R^p)_a{}^{b} \cdot \\ \cdot \langle \tau_\bullet(T^a) \rangle_{0,d-e} \cdot \langle \tau_\bullet(T_b) T^{b_j} \rangle^{N-1-j}_{e-e_j} \qquad (3)$$

has maximal rank N. This is what we will show in the next section. In fact, we will prove that *every* $N \times N$ submatrix of V is invertible. So we have shown

Theorem 2.1. *The Virasoro conditions together with the topological recursion relations of corollary 1.10 give a constructive way to determine all Gromov-Witten invariants of projective spaces.*

In contrast to other known relations that in theory determine the Gromov-Witten invariants (see [GP], [LLY]), our algorithm is easily implemented on a computer. No complicated sums over graphs occur anywhere in the procedure. It should be noted however that the calculation of some genus-g degree-d invariant usually requires the recursive calculation of invariants of smaller genus with *bigger* degree and *more* marked points. This is the main factor for slowing down the algorithm as the genus grows.

Some numbers that have been computed using this algorithm can be found in section 4.

3. Computation of the determinant

The goal of this section is to prove the technical result needed for theorem 2.1:

Proposition 3.1. *Fix any $N \geq 1$, and let $V^{(N)} = (V^{(N)}_{d,j})_{d\geq\delta,0\leq j<N}$ be the matrix defined in equation (3). Then any $N \times N$ submatrix of $V^{(N)}$, obtained by picking N distinct values of d, has non-zero determinant.*

We will prove this statement in several steps. In a first step, we will make the entries of the matrix independent of N and reduce the invariants $\langle\cdots\rangle^i$ to ordinary rational Gromov-Witten invariants:

Lemma 3.2. *Let $W = (W_{d,j})_{d\geq\delta,j\geq 0}$ be the matrix with entries $W_{d,j} = V^{(j+1)}_{d,j}$. Then:*

(i) *For all $d \geq \delta$, $N \geq 1$, and $0 \leq j < N$ we have*

$$V^{(N+1)}_{d,j} = V^{(N)}_{d,j} - \langle T_{b_N} T^{b_j}\rangle^{N-1-j}_{e_N-e_j} \cdot W_{d,N}.$$

(ii) *For any $N \geq 1$ and any $N \times N$ submatrix of W obtained by taking the* first *N columns of* any *N rows, the determinant of this submatrix is the same as the corresponding submatrix of $V^{(N)}$.*

Proof. (i): Comparing the q^{e-e_j}-terms of the recursive relations of corollary 1.6 we find that

$$\langle\tau_\bullet(T_b)T^{b_j}\rangle^{N-j}_{e-e_j} = \langle\tau_\bullet(T_b)T^{b_j}\rangle^{N-1-j}_{e-e_j} - \langle\tau_\bullet(T_b)T^{b_N}\rangle_{0,e-e_N}\langle T_{b_N}T^{b_j}\rangle^{N-1-j}_{e_N-e_j},$$

from which the claim follows.

(ii): We prove the statement by induction on N. There is nothing to show for $N = 1$. Now assume that we know the statement for some value of N, i.e. any two corresponding $N \times N$ submatrices of the matrices with columns

$$(W_{\cdot,0}, \ldots, W_{\cdot,N-1}) \quad \text{and} \quad (V^{(N)}_{\cdot,0}, \ldots, V^{(N)}_{\cdot,N-1})$$

have the same determinant. Of course, the same is then also true for any corresponding $(N + 1) \times (N + 1)$ submatrices of

$$(W_{\cdot,0}, \ldots, W_{\cdot,N-1}, W_{\cdot,N}) \quad \text{and} \quad (V^{(N)}_{\cdot,0}, \ldots, V^{(N)}_{\cdot,N-1}, W_{\cdot,N}).$$

But by (i), the latter matrix is obtained from

$$(V^{(N+1)}_{\cdot,0}, \ldots, V^{(N+1)}_{\cdot,N-1}, W_{\cdot,N}) = (V^{(N+1)}_{\cdot,0}, \ldots, V^{(N+1)}_{\cdot,N})$$

by an elementary column operation, so the result follows. □

So by the lemma, it suffices to consider the matrix W. Let us now evaluate the genus-0 Gromov-Witten invariants contained in the definition of W.

Convention 3.3. *For the rest of this section, we will make the usual convention that a product $\prod_{i=i_1}^{i_2} A_i$ is defined to be $\prod_{i=i_2+1}^{i_1-1} A_i^{-1}$ if $i_1 > i_2$.*

Lemma 3.4. *For all $d \geq \delta$ and $j \geq 0$ the matrix entry $W_{d,j}$ is equal to the z^j-coefficient of*

$$-\frac{\prod_{i=0}^{k(d)}\left(r + 1 + \left(\frac{3-r}{2} + i\right)z\right)}{(1 + (d - e_j)z)^{b_j+1}\prod_{i=0}^{d-e_j-1}(1 + iz)^{r+1}}.$$

Proof. Recall that by equation (3) the matrix entries $W_{d,j}$ are given by

$$\sum_{e,p} \underbrace{\left[\frac{3-r}{2} - (d-e)(r+1)\right]_p^{k(d)} (R^p)_a{}^b}_{\text{(A)}} \underbrace{(-1)^{1-a-(d-e)(r+1)} \langle \tau_\bullet(T^a) \rangle_{0,d-e}}_{\text{(B)}} \underbrace{\langle \tau_\bullet(T_b) T^{b_j} \rangle_{0,e-e_j}}_{\text{(C)}} .$$

The three terms in this expression can all be expressed easily in terms of generating functions. Recalling that $(R^p)_a{}^b = (r+1)^p \delta_{a+p,b}$, the (A) term is by definition equal to the z^p-coefficient of

$$\delta_{a+p,b} \prod_{i=0}^{k(d)} \left((r+1)z + \frac{3-r}{2} - (d-e)(r+1) + i \right).$$

The (B) and (C) terms are rational 2-point invariants of $\mathbb{P}^r$ which have been computed in [P] section 1.4: the Gromov-Witten invariant in (B) (without the sign) is equal to the z^a-coefficient of $\prod_{i=1}^{d-e} \frac{1}{(z+i)^{r+1}}$. So including the sign factor we get the z^a-coefficient of $-\prod_{i=1}^{d-e} \frac{1}{(z-i)^{r+1}}$. The (C) term is again by [P] equal to the z^{r-b}-coefficient of $\frac{1}{(z+e-e_j)^{b_j+1}} \prod_{i=1}^{e-e_j-1} \frac{1}{(z+i)^{r+1}}$.

Multiplying these expressions and performing the sums over a, b, and p, we find that $W_{d,j}$ is the z^r-coefficient of

$$-\sum_e \frac{\prod_{i=0}^{k(d)} \left((r+1)z + \frac{3-r}{2} - (d-e)(r+1) + i \right)}{(z+e-e_j)^{b_j+1} \prod_{i=1}^{d-e} (z-i)^{r+1} \cdot \prod_{i=1}^{e-e_j-1} (z+i)^{r+1}},$$

which can be rewritten as the sum of residues

$$-\sum_e \operatorname{res}_{z=0} \frac{\prod_{i=0}^{k(d)} \left((r+1)z + \frac{3-r}{2} - (d-e)(r+1) + i \right)}{(z+e-e_j)^{b_j+1} \prod_{i=e-d}^{e-e_j-1} (z+i)^{r+1}} \, dz.$$

Note that this fraction depends on z and e only in the combination $z+e$. Consequently, instead of summing the above residues at 0 over all e we can as well set $e = d$ and sum over all poles $z \in \mathbb{C}$ of the rational function. So we see that $W_{d,j}$ is equal to

$$-\sum_{z_0 \in \mathbb{C}} \operatorname{res}_{z=z_0} \frac{\prod_{i=0}^{k(d)} \left((r+1)z + \frac{3-r}{2} + i \right)}{(z+d-e_j)^{b_j+1} \prod_{i=0}^{d-e_j-1} (z+i)^{r+1}} \, dz.$$

By the residue theorem this is nothing but the residue at infinity of our rational function. So we conclude that

$$W_{d,j} = \operatorname{res}_{z=0} \frac{\prod_{i=0}^{k(d)} \left(\frac{r+1}{z} + \frac{3-r}{2} + i \right)}{(\frac{1}{z} + d - e_j)^{b_j+1} \prod_{i=0}^{d-e_j-1} (\frac{1}{z} + i)^{r+1}} \, d(\frac{1}{z}).$$

Finally note that by equations (1) and (2) we have the dimension condition

$$k(d) + 1 = j + b_j + (d-e_j)(r+1),$$

so multiplying our expression with $z^{k(d)+1}$ in the numerator and denominator we get

$$W_{d,j} = -\operatorname{res}_{z=0} \frac{\prod_{i=0}^{k(d)} \left(r+1 + \left(\frac{3-r}{2} + i \right) z \right)}{z^{j+1} (1+(d-e_j)z)^{b_j+1} \prod_{i=0}^{d-e_j-1} (1+iz)^{r+1}} \, dz.$$

This proves the lemma. □

To avoid unnecessary factors in the determinants, let us divide row d of W by the non-zero number $-(r+1)^{k(d)+1}$ and call the resulting matrix $\tilde{W}$. So we will now consider $N \times N$ submatrices of $\tilde{W} = (\tilde{W}_{d,j})$, obtained by picking the first N columns of any N rows, where $\tilde{W}_{d,j}$ is the z^j-coefficient of

$$\frac{\prod_{i=0}^{k(d)} \left(1 + \left(\frac{3-r}{2r+2} + \frac{i}{r+1}\right) z\right)}{(1+(d-e_j)z)^{b_j+1} \prod_{i=0}^{d-e_j-1} (1+iz)^{r+1}}. \tag{4}$$

The following technical lemma is the main step in computing their determinants.

Lemma 3.5. *Assume that we are given $N, n \in \mathbb{N}$, $M \in \mathbb{Z}$, $q, c \in \mathbb{R}$, and distinct integers $a_0, \ldots, a_N$. Set*

$$f(z) = \sum_{k=0}^{N} \left(\prod_{i \neq k} \frac{1}{a_k - a_i} \cdot \prod_{i=M}^{na_k} \left(1 + \frac{c+i}{n} z\right) \cdot (1+a_k z)^q \cdot \prod_{i=a_k}^{-1} (1+iz)^n \right)$$

as a formal power series in z.

(i) *For any $i \geq 0$ the z^i-coefficient of $f(z)$ is a symmetric polynomial in $a_0, \ldots, a_N$ of degree at most $i - N$. (In particular, it is zero for $i < N$.)*

(ii) *The z^N-coefficient of $f(z)$ is equal to*

$$\frac{1}{N!} \prod_{i=1}^{N} \left(c + q - N + \frac{n+1}{2} + i\right).$$

Proof. In the following proof, we will slightly abuse notation and vary the arguments given explicitly for the function f. So if we e.g. want to study how $f(z)$ changes if we vary c, we will write $f(z)$ also as $f(z,c)$, and denote by $f(z,c+1)$ the function obtained from $f(z)$ when substituting c by $c+1$.

(i): It is obvious by definition that $f(z)$ is symmetric in the a_i. We will prove the polynomiality and degree statements by induction on N.

"$N = 0$": In this case we have

$$f(z) = \prod_{i=M}^{na_0} \left(1 + \frac{c+i}{n} z\right) \cdot (1+a_0 z)^q \cdot \prod_{i=a_0}^{-1} (1+iz)^n.$$

We have to show that the z^i-coefficient of $f(z)$ is a polynomial in a_0 of degree at most i. Note that this property is stable under taking products, so if we write $f(z) = \prod_{j=0}^{n} f^{(j)}(z)$ with

$$f^{(0)}(z) = \prod_{i=M}^{0} \left(1 + \frac{c+i}{n} z\right) \cdot (1+a_0 z)^q$$

$$\text{and} \quad f^{(j)}(z) = \prod_{i=0}^{a_0-1} \left(1 + \frac{c+j}{n} \frac{z}{1+iz}\right) \quad \text{for } 1 \leq j \leq n$$

then it suffices to prove the statements for the $f^{(j)}$ separately. But the statement is obvious for $f^{(0)}$, so let us focus on $f^{(j)}$ for $j > 0$. Note that

$$f^{(j)}(z, a_0+1) = f^{(j)}(z, a_0) \cdot \left(1 + \frac{c+j}{n} \frac{z}{1+a_0 z}\right).$$

So if f_i denotes the z^i-coefficient of $f(z)$ we get

$$f_i^{(j)}(a_0+1) - f_i^{(j)}(a_0) = \frac{c+j}{n} \sum_{k=0}^{i-1} (-a_0)^k f_{i-1-k}^{(j)}(a_0). \tag{5}$$

The statement now follows by induction on i: it is obvious that the constant z-term of $f^{(j)}(z)$ is 1. For the induction step, assume that we know that $f_i^{(j)}$ is polynomial of degree at most i in a_0 for $i = 0, \dots, i_0 - 1$. Then the right hand side of (5) is polynomial of degree at most $i_0 - 1$ in a_0, so $f_{i_0}^{(j)}$ is polynomial of degree at most i_0. This completes the proof of the $N = 0$ part of (i).

"$N \to N+1$": Note that

$$f(z, N+1, a_0, \dots, a_{N+1}) = \frac{f(z, N, a_0, \dots, a_N) - f(z, N, a_1, \dots, a_{N+1})}{a_0 - a_{N+1}}. \tag{6}$$

By symmetry we have $f(z, N, a_0, \dots, a_N) = f(z, N, a_1, \dots, a_{N+1})$ if $a_0 = a_{N+1}$. Hence every z^i-coefficient of this expression is a polynomial in the a_k. Its degree is at most $(i - N) - 1$ by the induction hypothesis. This proves (i).

(ii): By (i) the z^N-coefficient of $f(z, N)$ does not depend on the choice of a_k, so we can set $a_k = a + k$ for all k and keep only a as a variable. It does not depend on M either, as a shift $M \mapsto M \pm 1$ corresponds to multiplication of $f(z)$ with $(1 + \alpha z)^{\mp 1}$ for some α, which does not affect the leading coefficient of $f(z)$. So we can set $M = 1$ without loss of generality.

The recursion relation (6) now reads

$$f(z, N+1, a) = \frac{f(z, N, a+1) - f(z, N, a)}{N+1}. \tag{7}$$

By (i) the z^i-coefficient of $f(z)$ has degree at most $i - N$ in a. So if we denote by f_i the a^{i-N}-coefficient of the z^i-coefficient of $f(z, a)$, comparing the z^i-coefficients in (7) yields $f_i(N+1) = \frac{i-N}{N+1} f_i(N)$ and therefore

$$f_N(N) = \frac{1}{n} \cdot \frac{2}{n-1} \cdots \frac{n}{1} \cdot f_N(0) = f_N(0).$$

In other words, instead of computing the z^N-coefficient of $f(z, N)$ we can as well set $N = 0$ and compute the a^N-coefficient (i.e. the leading coefficient in a) of the z^N-coefficient of $f(z, N = 0)$. So let us set $N = 0$ to obtain

$$f(z) = \prod_{i=1}^{na} \left(1 + \frac{c+i}{n} z\right) \cdot (1 + az)^q \cdot \prod_{i=a}^{-1} (1 + iz)^n,$$

and denote by g_N the a^N-coefficient of the z^N-coefficient of $f(z, a)$. Moreover, set $g(z) = \sum_{N \geq 0} g_N z^N$. Our goal is then to compute $g(z)$.

We will do this by analyzing how $f(z)$ (and thus $g(z)$) varies when we vary q, c, or n. To start, it is obvious that

$$g(z, q+\alpha) = (1+z)^\alpha g(z, q) \tag{8}$$

for all $\alpha \in \mathbb{R}$. Next, note that

$$f(z, c+1) = f(z, c) \cdot \frac{1 + \frac{c+1+na}{n} z}{1 + \frac{c+1}{n} z}.$$

For $g(z)$ we can drop all terms in which the degree in a is smaller than the degree in z. So we conclude

$$g(z,c+1) = (1+z)g(z,c).$$

Combining this with (8) we see that $g(z)$ will depend on q and c only through their sum $q+c$. So in what follows we can set $c=0$, and replace q by $q+c$ in the final result.

Varying n is more complicated. We have

$$f(z,n+1) = f(z,n) \cdot \underbrace{\prod_{j=0}^{n}\prod_{i=1}^{a}\left(1+\frac{j}{n(n+1)}\cdot\frac{z}{1+(i-\frac{j}{n})z}\right)}_{=:\tilde{f}(z)}.$$

Recall that for $g(z)$ we only need the summands in $\tilde{f}(z)$ in which the degree in a is equal to the degree in z. So let us denote the a^N-coefficient of the z^N-coefficient of $\tilde{f}(z,a)$ by $\tilde{g}_N$, and assemble the $\tilde{g}_N$ into a generating function $\tilde{g}(z) = \sum_{N\geq 0}\tilde{g}_N z^N$, so that $g(z,n+1) = g(z,n)\cdot\tilde{g}(z)$. To determine $\tilde{g}(z)$ compare the a^N-coefficient of the z^N-coefficient in the recursive equation

$$\frac{\tilde{f}(z,a)-\tilde{f}(z,a-1)}{z} = \tilde{f}(z,a-1)\cdot\frac{1}{z}\left(\prod_{j=0}^{n}\left(1+\frac{j}{n(n+1)}\cdot\frac{z}{1+(a-\frac{j}{n})z}\right)-1\right).$$

On the left hand side this coefficient is $(N+1)\tilde{g}_{N+1}$. On the right hand side it is the z^N-coefficient of

$$\tilde{g}(z)\cdot\left(\sum_{j=0}^{N}\frac{j}{n(n+1)}\right)\cdot\frac{1}{1+z} = \tilde{g}(z)\cdot\frac{1}{2}\,\frac{1}{1+z}.$$

So we see that

$$\frac{d\tilde{g}(z)}{dz} = \frac{1}{2}\,\frac{1}{1+z}\,\tilde{g}(z).$$

Together with the obvious initial condition $\tilde{g}(0) = 1$ we conclude that $\tilde{g}(z) = \sqrt{1+z}$, and therefore

$$g(z,n+1) = g(z,n)\cdot\sqrt{1+z}.$$

Comparing this with (8) we see that $g(z)$ depends on n and q only through the sum $q+\frac{n}{2}$. We can therefore set $n=1$ and then replace q by $q+\frac{n-1}{2}$ in the final result.

But setting n to 1 (and c to 0) we are simply left with

$$f(z) = \prod_{i=1}^{a}(1+iz)\cdot(1+az)^q\cdot\prod_{i=a}^{-1}(1+iz) = (1+az)^{q+1}.$$

So it follows that $g(z) = (1+z)^{q+1}$ and therefore

$$g_N = \binom{q+1}{N} = \frac{1}{N!}\prod_{i=1}^{N}(q+1-N+i).$$

Setting back in the c and n dependence, i.e. replacing q by $q+c+\frac{n-1}{2}$, we get the desired result. □

We are now ready to compute our determinant.

Proposition 3.6. *Let $(\tilde{W})_{d\geq\delta,j\geq 0}$ be the matrix defined in equation (4). Pick N distinct integers $d_0,\dots,d_{N-1}$ with $d_i \geq \delta$ for all i. Then the determinant of the $N\times N$ submatrix of $\tilde{W}$ obtained by picking the first N columns of rows $d_0,\dots,d_{N-1}$ is equal to*

$$\frac{\prod_{i>j}(d_i-d_j)}{\prod_{i=1}^{N-1} i!} \cdot \prod_{i=1}^{N-1} \left(i+\frac{1}{2}\right)^{N-i}.$$

In particular, this determinant is never zero.

Proof. We prove the statement by induction on N. The result is obvious for $N=1$ as every entry in the first column (i.e. $j=0$) of $\tilde{W}$ is equal to 1. So let us assume that we know the statement for a given value N. We will prove it for $N+1$.

Denote by $\Delta(d_0,\dots,d_{N-1})$ the determinant of the $N\times N$ submatrix of $\tilde{W}$ obtained by picking the first N columns of rows $d_0,\dots,d_{N-1}$. Then by expansion along the last column and the induction assumption we get

$$\begin{aligned}\Delta(d_0,\dots,d_N) &= \sum_{k=0}^{N}(-1)^{k+N}\tilde{W}_{d_k,N}\cdot\Delta(d_0,\dots,d_{k-1},d_{k+1},\dots,d_N) \\ &= \frac{\prod_{i>j}(d_i-d_j)}{\prod_{i=1}^{N-1} i!} \cdot \prod_{i=1}^{N-1}\left(i+\frac{1}{2}\right)^{N-i} \cdot \sum_{k=0}^{N}\left(\prod_{i\neq k}\frac{1}{d_k-d_i}\cdot\tilde{W}_{d_k,N}\right).\end{aligned}$$

But by lemma 3.5, applied to the values $n=r+1$, $a_k=d_k-e_N$, $q=-b_N-1$, $M=(r+1)(d_k-e_N)-k(d_k)=1-N-b_N$, and $c=\frac{3-r}{2}-M=\frac{1-r}{2}+N+b_N$, the sum in this expression is equal to

$$\frac{1}{N!}\prod_{i=1}^{N}\left(i+\frac{1}{2}\right).$$

Inserting this into the expression for the determinant, we obtain

$$\Delta(d_0,\dots,d_N) = \frac{\prod_{i>j}(d_i-d_j)}{\prod_{i=1}^{N} i!}\cdot\prod_{i=1}^{N}\left(i+\frac{1}{2}\right)^{N+1-i},$$

as desired. □

Remark 3.7. It should be remarked that the expression for the determinant in proposition 3.6 is surprisingly simple, given the complicated structure of the Virasoro conditions and the topological recursion relations. It would be interesting to see if there is a deeper relation between these two sets of equations that is not yet understood and explains the simplicity of our results.

Combining the arguments of this section, we see that the systems of linear equations obtained from the Virasoro conditions and the topological recursion relations in section 2 are always solvable. This completes the proof of theorem 2.1.

4. Some numbers

In this section we will give some examples of invariants that have been computed using the method of this paper.

Example 4.1. (The Caporaso-Harris numbers, see [CH]) The following table shows some numbers of curves in $\mathbb{P}^2$ of genus g and degree d through $3d-1+g$ points, i.e. the Gromov-Witten invariants $\langle \mathrm{pt}^{3d-1+g} \rangle_{g,d}$. They can be found either by the Caporaso-Harris method or by applying the techniques of this paper.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$g=0$	1	1	12	620	87304	26312976	14616808192
$g=1$	0	0	1	225	87192	57435240	60478511040
$g=2$	0	0	0	27	36855	58444767	122824720116
$g=3$	0	0	0	1	7915	34435125	153796445095
$g=4$	0	0	0	0	882	12587820	128618514477

Example 4.2. We list some 1-point invariants of $\mathbb{P}^2$, i.e. invariants of the form $\langle \tau_m(\gamma) \rangle_{g,d}$, where m is determined by the dimension condition $3d+g = m+\deg\gamma$.

	$\gamma=\mathrm{pt}$			$\gamma=H$			$\gamma=1$		
	$d=0$	$d=1$	$d=2$	$d=0$	$d=1$	$d=2$	$d=0$	$d=1$	$d=2$
$g=0$	–	1	$\frac{1}{8}$	–	-3	$-\frac{9}{16}$	–	6	$\frac{3}{2}$
$g=1$	–	0	$\frac{1}{32}$	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{3}{32}$	$\frac{1}{8}$	$-\frac{1}{4}$	$\frac{23}{128}$
$g=2$	0	$-\frac{1}{240}$	$-\frac{1}{960}$	$-\frac{1}{960}$	$-\frac{1}{960}$	$\frac{13}{1536}$	$\frac{7}{640}$	$\frac{1}{128}$	$-\frac{27}{1280}$
$g=3$	0	$\frac{1}{3360}$	$-\frac{1}{16128}$	$-\frac{1}{40320}$	0	$-\frac{163}{645120}$	$\frac{41}{161280}$	$-\frac{97}{161280}$	$\frac{43}{36864}$
$g=4$	0	$-\frac{1}{80640}$	$\frac{11}{1075200}$	$-\frac{1}{1075200}$	$-\frac{1}{153600}$	$-\frac{1}{147456}$	$\frac{127}{12902400}$	$\frac{173}{4300800}$	$-\frac{4567}{103219200}$

Example 4.3. The following table gives some Gromov-Witten invariants $\langle \mathrm{pt}^{2d} \rangle_{g,d}$ of $\mathbb{P}^3$, i.e. the virtual number of genus-g degree-d curves in $\mathbb{P}^3$ through $2d$ points. Note that these Gromov-Witten invariants are not enumerative for $g > 0$. The precise relationship between the Gromov-Witten invariants and the enumerative numbers is not yet known.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$g=0$	1	0	1	4	105	2576	122129
$g=1$	$-\frac{1}{12}$	0	$-\frac{5}{12}$	$-\frac{4}{3}$	$-\frac{147}{4}$	$\frac{1496}{3}$	$\frac{1121131}{12}$
$g=2$	$\frac{1}{360}$	0	$\frac{1}{12}$	$-\frac{1}{180}$	$-\frac{49}{8}$	$-\frac{7427}{5}$	$-\frac{4905131}{45}$
$g=3$	$-\frac{1}{20160}$	0	$-\frac{43}{4032}$	$\frac{103}{1080}$	$\frac{473}{64}$	$\frac{206873}{270}$	$\frac{283305113}{8640}$
$g=4$	$\frac{1}{1814400}$	0	$\frac{713}{725760}$	$-\frac{26813}{907200}$	$-\frac{833}{320}$	$-\frac{12355247}{56700}$	$-\frac{1332337}{34560}$

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